DYBAC Memento Pt. I

(1) Introduction

$$\begin{bmatrix}\dot{x}\\ y\end{bmatrix}=\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}+\begin{bmatrix}\text{Dim}(X)=n\\ \text{Dim}(U)=m\\ \text{Dim}(Y)=p\end{bmatrix}\xrightarrow{\mathcal{L}}\begin{bmatrix}sX(s)-x_\emptyset\\ Y(s)\end{bmatrix}\longrightarrow\left[Y(s)=\underbrace{\overbrace{\left[C(sI-A)^{-1}B+D\right]}^{\text{TRANSFER FUNCTION } H(s)}U(s)}_{\text{FORCED RESPONSE}}+\underbrace{(sI-A)^{-1}x_\emptyset}_{\text{FREE RESPONSE}}\right]$$

$$\mathcal{L}\{e^{At}\}=\Phi(s)=(sI-A)^{-1}$$

$$\left[\begin{aligned}x(t)&=e^{A(t-t_\emptyset)}x(t_\emptyset)+\int_{t_\emptyset}^{t}e^{A(t-\tau)}Bu(\tau)\,d\tau\\ y(t)&=Ce^{A(t-t_\emptyset)}x(t_\emptyset)+\int_{t_\emptyset}^{t}Ce^{A(t-\tau)}Bu(\tau)\,d\tau+Du(t)\end{aligned}\right]$$

$\begin{bmatrix}\text{RIGHT}\\ \text{EIGENV.}\end{bmatrix}$ $Av=\lambda v$ $\quad\begin{bmatrix}\text{LEFT}\\ \text{EIGENV.}\end{bmatrix}$ $w^TA=w^T\lambda$

↳ IF $X(\emptyset)=v\Rightarrow(\dot{x}=Ax)$ SOLUTION IS $x(t)=\underbrace{e^{\lambda t}\cdot v}_{\text{(MODE OF THE SYS.)}}$

$$\left[y(t)=\left[\underbrace{Ce^{A(t)}B+D\delta(t)}_{\text{IMPULSE RESPONSE } h(t)=\mathcal{L}^{-1}\{H(s)\}}\right]*u(t)\right]$$

(FOR THE LEFT!)

↳ IF $x(t)=e^{At}x(\emptyset)$ IS SOLUTION OF $(\dot{x}=Ax)\Rightarrow w^Tx(t)=e^{\lambda t}w^Tx(\emptyset)$

(IS A WAY TO EXTRACT THIS MODE)

$(w_i^Tv_j=\delta_{ij})$

→ [A DIAGONALIZABLE] ⟺ $[V=\{v_i,\forall i\}$ ALL v_i WITH DIFFERENT $\lambda_i]$

↳ [ALL λ_i HAVE MULTIPLICITY 1]

HOW TO DIAGONALIZE: $T=[v_1\cdots v_n]$, $T^{-1}=\begin{bmatrix}w_1^T\\ \vdots\\ w_n^T\end{bmatrix}$ we can do: $T^{-1}AT=\Delta=\text{diag}(\lambda_1,\dots,\lambda_n)$

→ IF $Cv_i=\emptyset$ THE MODE (λ_i,v_i) IS UNOBSERVABLE

→ IF $w_i^TB=\emptyset$ THE MODE (λ_i,v_i) IS UNCONTROLLABLE

INTERNAL STABILITY ⟺ $\text{Re}\{\lambda_i\}<\emptyset,\ \forall i$

BIBO (EXTERNAL) STABILITY ⟺ $\lim_{t\to\infty}h(t)=\emptyset\iff\text{Re}\{p_i\}<\emptyset\ \forall i$

[POLES OF $H(s)$]

(2) Controllability

$$\mathcal{C}=[B|AB|\cdots|A^{n-1}B]\qquad[\text{CONTROLLABLE}\iff\text{rank}(\mathcal{C})=n]$$

(IF NOT)

$$T=\Big[\overbrace{R_\emptyset=\text{Im}(\mathcal{C})}^{\tilde{x}_1}\ \Big|\ \overbrace{\text{COMPLETION}}^{\tilde{x}_2}\Big]\qquad \mathcal{X}=\tilde{\mathcal{X}}_1\oplus\tilde{\mathcal{X}}_2\ \text{(REACHABLE STATES SPACE)}\ R_\emptyset$$

$$T^{-1}AT=\begin{bmatrix}A_{11} & A_{12}\\ \emptyset & A_{22}\end{bmatrix}\qquad T^{-1}B=\begin{bmatrix}B_1\\ \emptyset\end{bmatrix}$$

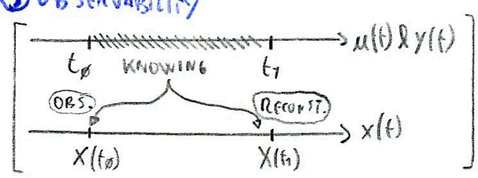
(IS THE VECTOR SPACE GENERATED BY $\mathcal{C}$
↳ I NEED A BASE OF $\mathcal{C}$ HERE) (THE BASE VECTORS MISSING IN $\mathcal{C}$)

$\begin{bmatrix}\text{UNCONTROLLABLE}\\ \text{MODES}\end{bmatrix}$: $(w^TA=w^T\lambda)$ AND $(w^TB=\emptyset)$

(TRANSFORM $x=T\tilde{x}$, $u=G\tilde{u}$) ⇒ $\text{rank}(\mathcal{C})=\text{rank}(T^{-1}\mathcal{C}G)$

[ON $\tilde{\mathcal{X}}_2$ THE SYS IS REDUCED TO $\dot{\tilde{x}}_2=A_{22}\tilde{x}_2$ & NO CONTROL ACTION!
↳ UNCONTROLLABLE PART OF THE SYS
ON $\tilde{\mathcal{X}}_1$: $R_0=\text{Im}(\mathcal{C})$ IS THE SET OF REACHABLE STATE
IF INITIAL CONDITION IS IN $R_\emptyset$, THE GLOBAL TRAJECTORY WILL BE IN $R_\emptyset$
↳ CONTROLLABLE PART OF THE SYSTEM]

(3) Observability



OBSERVABILITY ⇓ ($\cancel{⇑}$ NOT FOR DISCRETE SYS) RECONSTRUCTIBILITY

$$\mathcal{O}=\begin{bmatrix}C\\ CA\\ \vdots\\ CA^{n-1}\end{bmatrix}\qquad\text{rank}(\mathcal{O})=n\iff\text{OBSERVABLE}$$

$$\left.\begin{aligned}&N_\emptyset\ \text{UNOBSERVABLE SUBSPACE}=\bigcap_{i=1}^{n}\text{Ker}(CA^{i-1})\\ &R_0=\text{Im}(\mathcal{C})\ \text{CONTROLLABLE SUBSPACE}=\sum_{i=1}^{n}\text{im}(A^{i-1}B)\end{aligned}\right\}\ \text{DUALITY!}\quad\bigcap_{i=1}^{n}\text{Ker}(CA^{i-1})=\emptyset\iff\sum_{i=1}^{n}\text{im}(A^T)^{i-1}C^T=\mathcal{X}$$

(C,A) PAIR OBSERVABLE ⟺ (A^T,C^T) CONTROLLABLE $\left(\rightarrow\begin{cases}\dot{x}^d=A^Tx^d+C^Tu^d\\ y^d=B^Tx^d+D^Tu^d\end{cases}\right)$

(IF NOT OBSERVABLE)

$$T=\Big[\underbrace{\text{COMPLETION}}_{\bar{\mathcal{X}}_1}\ \Big|\ \underbrace{N_\emptyset=\text{Ker}(\mathcal{O})}_{\bar{\mathcal{X}}_2}\Big]\qquad\mathcal{X}=\bar{\mathcal{X}}_1\oplus\bar{\mathcal{X}}_2\ \text{(UNOBSERVABLE STATE SPACE)}\ N_\emptyset$$

$$\rightarrow T^{-1}AT=\begin{bmatrix}A_{11} & \emptyset\\ A_{21} & A_{22}\end{bmatrix}\qquad CT=[C_1|\emptyset]$$

$\begin{bmatrix}\text{UNOBSERVABLE}\\ \text{MODES}\end{bmatrix}$: $(Av=\lambda v)$ AND $(Cv=\emptyset)$

(TRANSFORM $x=T\bar{x}$, $y=S\bar{y}$) ⇒ $\text{rank}(\mathcal{O})=\text{rank}(S^{-1}\mathcal{O}T)$

[ON $\bar{\mathcal{X}}_2$: $AN_\emptyset\subset N_\emptyset$: ALL TRAJECTORIES STARTED IN $N_\emptyset$ STAY IN $N_\emptyset$
THE OUTPUT IS INDEPENDENT OF THE STATE OF THIS PART
↳ UNOBSERVABLE PART OF THE SYS.
ON $\bar{\mathcal{X}}_1$: THIS IS THE OBSERVABLE PART]

$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_3 + \mathcal{X}_4$

$\mathcal{X}_1$: $\boxed{C \wedge \neg O}$, $\mathcal{X}_2$: $\boxed{C \wedge O}$, $\mathcal{X}_3$: $\boxed{\neg C \wedge \neg O}$, $\mathcal{X}_4$: $\boxed{\neg C \wedge O}$

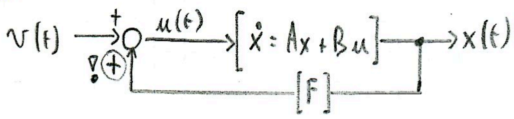
(COMPLETE DECOMPOSITION) ?

$$\left[\; T^{-1} A T = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \quad T^{-1} B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix} \quad CT = \begin{bmatrix} \emptyset & C_2 & 0 & C_4 \end{bmatrix} \quad D \;\right] \rightarrow H(s) = C_2 (sI - A_{22})^{-1} B_2 + D$$

(POLES ARE ONLY THE $C \wedge O$ EIGENVALUES!)

$$\left[\begin{cases} \dot{x}_2 = A_{22} x_2 + B u \\ y = C_2 x_2 + D u \end{cases} \leftarrow \text{[MINIMAL REALISATION]}\right]$$

(4) STATIC STATE SPACE FEEDBACK



(FOR A CONTROLLABILITY DECOMPOSED SYS?)

$$A + BF = \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\ \emptyset & A_{22} \end{bmatrix}$$

$(u = Fx + v) \rightarrow \dot{x} = (A + BF)x + Bv$

ON MATLAB: $\left[\begin{array}{l} \text{SISO: } F = -\text{acker}(A, B, \text{Poles}) \\ \text{MIMO: } F = -\text{place}(A, B, \text{Poles}) \end{array}\right]$

IF SYS STABILIZABLE, THEN $\rightarrow \det(sI - (A+BF)) = (s - \lambda d_1)(s - \lambda d_2)\cdots(s - \lambda d_m)$ ← [FOR Poles = $\{\lambda d_1, \dots, \lambda d_m\}$]

(5) OBSERVER STATE ESTIMATION FEEDBACK

- DIFFERENTIAL OBSERVERS:

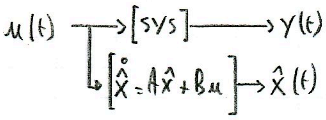
$\mathcal{O}$ TOEPLIZ MATRIX $\mathcal{T}^e$ → IF $\text{rank}(\mathcal{O}) = n \rightarrow \mathcal{O}$ PSEUDO INVERTIBLE ?

$$\begin{bmatrix} y^{(0)}(t) = y(t) \\ y^{(1)}(t) = d/dt\, y(t) \\ \vdots \\ y^{(m)}(t) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t) + \begin{bmatrix} \emptyset & \cdots & \emptyset \\ CB & \emptyset & \vdots \\ \vdots & CB & \emptyset \\ CA^{n-2}B & \vdots & CB \end{bmatrix} \begin{bmatrix} u^{(0)}(t) \\ u^{(1)}(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix} \Rightarrow x(t) = \mathcal{O}^{+} \left(\begin{bmatrix} y^{(0)} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T}^e \begin{bmatrix} u(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix} \right)$$

[ISTANTANEOUS OBSERVER]

(BUT INPUT AND OUTPUT DERIVATIVES ARE NOT PHYSICALLY REALIZ.)

- OPEN-LOOP OBSERVERS:



IF $(\hat{x}_\emptyset = \hat{x}(\emptyset) = x(\emptyset))$ then $(\hat{x}(t) = x(t) \ \forall t)$

↳ BUT HOW HAVE INFORMATION ABOUT $x_\emptyset$?

$$\rightsquigarrow x(\emptyset^-) = \mathcal{O}^{+} \begin{bmatrix} y(\emptyset^-) \\ \vdots \\ y^{(n-1)}(\emptyset^-) \end{bmatrix}$$

[ISTANTANEOUS OBSERVER]

[BUT IF SYS IS NOT IDEALLY IDENTIFIED THEN $\hat{x}_\emptyset = x_\emptyset - \varepsilon$]

↙ ERROR PROPAGATION!

ERROR PROPAGATION FOR OPEN LOOP OBSERVER:

$$(\tilde{x} \triangleq x - \hat{x}) \Rightarrow \begin{cases} \dot{\tilde{x}} = A\tilde{x} \\ \tilde{x}(\emptyset) = \varepsilon \end{cases}$$

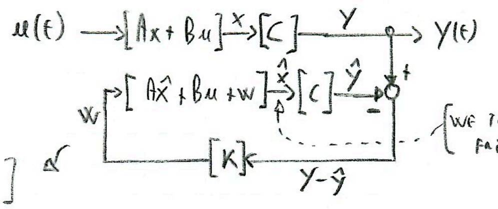
- IF (A IS STABLE): $\forall \tilde{x}(\emptyset)$, $\tilde{x} \xrightarrow{t \to \infty} \emptyset$ ASYMPTOTIC OBSERVER
  - ↳ BUT IF IT HAS TOO SLOW DYNAMICS IS NOT GOOD (NOT FAST ENOUGH)
- IF (A IS UNSTABLE): $\exists \tilde{x}(\emptyset)$, $\tilde{x} \xrightarrow[t \to \infty]{} \infty$ BAD!

→ FOR FASTEN THE DYNAMICS OR TO STABILIZE A YOU NEED TO MODIFY THE DYNAMICS OF $\dot{\tilde{x}} = A\tilde{x}$

- CLOSED LOOP OBSERVERS:

$(\dot{\tilde{x}} = A\tilde{x}) \xrightarrow{\text{MODIFY}} \dot{\tilde{x}} = (A - KC)\tilde{x}$

[K s.t. $(A - KC)$ IS STABLE]



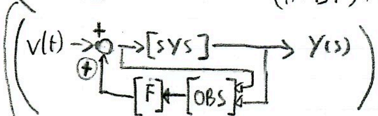
[WE TOOK $\hat{x}$ FROM HERE]

$[w = K(y - \hat{y})]$

[OBSERVER EQUATION (IF $D = \emptyset$): $\dot{\hat{x}} = (A - KC)\hat{x} + Bu + Ky$]

$$\begin{cases} \dot{\tilde{x}} = (A - KC)\tilde{x} \\ \tilde{x}(\emptyset) = x_\emptyset - \hat{x}_\emptyset \end{cases}$$

(IS DUAL OF FINDING $(A+BF)$ STABLE) ← [NOW THE PROBLEM IS TO FIND K S.T. $(A - KC)$ IS STABLE]



$$\rightarrow \left(\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A+BF & -BF \\ \emptyset & A-KC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ \emptyset \end{bmatrix} v \qquad y = \begin{bmatrix} C & \emptyset \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + D v \right)$$

→ $(A - KC)$ IS NOT CONTROLLABLE (ESTIMATION ERROR $\tilde{x}$ NOT CONTROLLABLE)

$H(s) = C(sI - (A+BF))^{-1} B$

$$y = \underbrace{H(s) \cdot v}_{\text{FORCED RESPONSE}} + \underbrace{C(sI - (A+BF))^{-1} \left[x(\emptyset) - BF(sI - (A-KC))^{-1} \tilde{x}(\emptyset) \right]}_{\text{INITIAL CONDITION RESPONSE} \ \xrightarrow[t \to \infty]{} \ \emptyset}$$

↳ POLE PLACEMENT:

$$\det(sI - (A - KC)) = (s - \lambda d_1)\cdots(s - \lambda d_m)$$

[OR ON MATLAB] → $\left[\begin{array}{l} \text{SISO } K = +\text{acker}(A', C', \text{poles}) \\ \text{MIMO } K = +\text{acker}(A', C', \text{poles}) \end{array}\right]$

• CANONICAL FORMS

$$H(s) = \frac{b_1 s^{m-1} + b_2 s^{m-2} + \dots + b_m}{\cancel{a_0} s^m + a_1 s^{m-1} + \dots + a_m} + d \qquad (a_0 = 1)$$

$\boxed{\tilde{b}_i = b_i - a_i b_\emptyset \ (\text{IF } \exists\, b_0 s^n \text{ IN NUMERATOR})}$

DUAL FORMS: $\left[\begin{array}{ll} A^d = A^T & B = C^T \\ C = B^T & D = D \end{array}\right]$

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_m \\ & I & & \begin{matrix} \emptyset \\ \vdots \\ \emptyset \end{matrix} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ \emptyset \\ \vdots \\ \emptyset \end{bmatrix} \quad C = \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_m \end{bmatrix} \quad D = d$$

(CONTROLLER) CONTROLLABLE CANONICAL FORM

$$A = \begin{bmatrix} [\emptyset \ \cdots \ \emptyset] & -a_n \\ I & \begin{matrix} -a_{n-1} \\ \vdots \\ -a_1 \end{matrix} \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_m \end{bmatrix} \quad D = d$$

CONTROLLABILITY CANONICAL FORM

DYBAC Memento Pt. II

① Point Dyn Principles (mass point M, A point ∈ E^3)

$\underline{P}_M = m\,\underline{V}_M$ [LINEAR MOMENTUM] ; [ANGULAR] $\underline{h}_M(A) = \vec{AM} \times \underline{P}_M = m\,\vec{AM} \times \underline{V}_M$ (N.E. LAW) → $\sum \underline{f} = \dot{\underline{P}}_M = m\,\dot{\underline{V}}_M$; $\sum \underline{m} : \dot{\underline{h}}_M(A) = m\,\vec{AM} \times \dot{\underline{V}}_M$

P. VIRTUAL POWERS for single rigid link $\mathcal{B}_J$: $P^*_{acc} = P^*_{ext}$ / virtual power due to inertial effects : $P^*_{acc} = m\,\underbrace{\dot{\underline{V}}_M^T}_{\text{(inertial effect on the particle)}}\,\underline{V}_M^*$

→ we can decompose $P^*_{ext} = P^*_{grav} + P^*_{reac}$ with $P^*_{grav} = m\underbrace{\underline{g}^T}_{\text{(}m\underline{g}\text{ gravity force on M)}}\underline{V}_M^*$ and $P^*_{reac} = \underbrace{\underline{f}^T}_{\text{(if there is a force on M)}}\underline{V}_M^*$

↓ S_J, since has to be valid ∀ v^*

$(\underline{f}^T + m\underline{g}^T)\,\underline{V}_M^* = (m\,\dot{\underline{V}}_M^T)\,\underline{V}_M^*$ → $\underline{f} + m\underline{g} = m\,\dot{\underline{V}}_M$ (EQUIVALENT TO N.E. LAW!)

LAGRANGE EQUATIONS : $\underline{\Upsilon} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\right)^T - \left(\frac{\partial L}{\partial \underline{q}}\right)^T$ ← (L = E − U) $\underline{V}_M = \dot{\underline{r}}_{OM} = \left[\frac{\partial \underline{r}_{OM}}{\partial \underline{q}}\right]\dot{\underline{q}} = \underline{\underline{J}}\,\dot{\underline{q}}$

$\left[\underline{\Upsilon}_q = \underline{\underline{J}}^T \sum \underline{f} = m\,\underline{\underline{J}}^T \dot{\underline{V}}_M\right]$ (generalized force) → $\left[\underline{\Upsilon}_q = \frac{d}{dt}\left(\frac{\partial E}{\partial \dot{\underline{q}}}\right)^T - \left(\frac{\partial E}{\partial \underline{q}}\right)^T\right]$ AND $\left[\underline{\Upsilon}_q = \underbrace{\underline{\Upsilon}_c}_{\text{(CONSERVATIVE FORCES)}} + \underline{\Upsilon}\right]$; $\left[\underline{\Upsilon}_c = -\left(\frac{\partial U}{\partial \underline{q}}\right)^T\right]$

[$\underline{\Upsilon}$ is NON CONSERVATIVE GENERALIZED FORCES VECTOR]

② Body Dyn Principles - Lagrange

FOR ANY RIGID ROBOT $\underline{\Upsilon} = \underbrace{\underline{\underline{M}}(\underline{q})}_{\text{(GENERALIZED INERTIA MATRIX)}}\ddot{\underline{q}} + \underbrace{\underline{C}(\underline{q}, \dot{\underline{q}})}_{\text{(VECTOR OF CORIOLIS, CENTRIFUGAL AND GRAVITY EFFECTS)}}$

[KINETIC ENERGY E_J OF A BODY $\mathcal{B}_J$] : $E_J = \frac{1}{2}\underline{t}_J^T \underline{\underline{M}}_J \underline{t}_J$; $\underline{\underline{M}}_J = \begin{bmatrix} m_J \cdot \mathbb{1}_3 & \hat{\underline{ms}}_J^T \\ \hat{\underline{ms}}_J^T & \underline{\underline{I}}_{O_J} \end{bmatrix}$

↓

$E_i = \frac{1}{2}\left(m_i\,{}^i\underline{V}_i^T\,{}^i\underline{V}_i + {}^i\underline{\omega}_i^T\,{}^i\underline{\underline{I}}_{O_i}\,{}^i\underline{\omega}_i + 2\,{}^i\underline{ms}_i^T({}^i\underline{V}_i \times {}^i\underline{\omega}_i)\right)$

$\underline{ms}_J = \begin{bmatrix} mx_J \\ my_J \\ mz_J \end{bmatrix} = m_J\,{}^J\underline{r}_{O_J S_J}$ ← (S_J is C.O. Mass of the Body J)

$\hat{\underline{ms}}_J = \begin{bmatrix} \emptyset & -mz_J & my_J \\ mz_J & \emptyset & -mx_J \\ -my_J & mx_J & \emptyset \end{bmatrix} = [\underline{ms}_J]_\times$ $\left(\underline{ms}_J = \int_{\mathcal{B}_J} {}^J\underline{r}_{O_J M_J}\,dm\right)$

$\underline{\underline{I}}_{O_J} = \int {}^J\hat{\underline{r}}_{O_J M_J} \cdot {}^J\hat{\underline{r}}_{O_J M_J}\,dm = \begin{bmatrix} xx_J & xy_J & xz_J \\ xy_J & yy_J & yz_J \\ xz_J & yz_J & zz_J \end{bmatrix}$ ← [INERTIA MATRIX OF BODY $\mathcal{B}_J$]

$\left(xx_J = \int_{\mathcal{B}_J}(r_y^2 + r_z^2)\,dm\right)$ $\left(xy_J = -\int_{\mathcal{B}_J} r_x r_y\,dm\right)$

VELOCITY OF A POINT $M \in \mathcal{B}_J$: ∂

$\underline{V}_{M_J} = \underline{V}_J + \underline{\omega}_J \times \underline{r}_{O_J M_J} = \left[\mathbb{1}_3 \;\; \hat{\underline{r}}_{O_J M_J}\right]\begin{bmatrix} \underline{V}_J \\ \underline{\omega}_J \end{bmatrix} = \left[\mathbb{1}_3 \;\; \hat{\underline{r}}_{O_J M_J}\right]\underline{t}_J$ (TWIST)

For the whole system: $\underline{\underline{M}}(\underline{q}) = \sum_J \left({}^J\underline{\underline{J}}_J^T(\underline{q}) \cdot \underline{\underline{M}}_J \cdot {}^J\underline{\underline{J}}_J(\underline{q})\right)$ [SYS INERTIA MATRIX] $E = \sum_J [E_J] = \frac{1}{2}\dot{\underline{q}}^T \underline{\underline{M}}(\underline{q})\dot{\underline{q}}$ (cause ${}^J\underline{t}_J = {}^J\underline{\underline{J}}_J\,\dot{\underline{q}}$)

[POTENTIAL ENERGY U_J OF BODY $\mathcal{B}_J$] → $U_J = -\left[{}^\emptyset\underline{g}^T, \emptyset\right]{}^\emptyset\underline{\underline{T}}_i\begin{bmatrix} {}^i\underline{ms}_i \\ m_i \end{bmatrix}$ WITH ${}^\emptyset\underline{\underline{T}}_i = \begin{bmatrix} [{}^\emptyset\underline{\underline{R}}_i] & [\vec{O_\emptyset O_i}] \\ \emptyset\emptyset\emptyset & 1 \end{bmatrix}$ $U = \sum_J [U_J]$

FOR CLOSED LOOP MECHANISMS : [] → CONSTRAINT RELATIONSHIPS → $\left[\begin{array}{l} \underline{h}(\underline{q}_a, \underline{q}_d) = \emptyset \text{ (active) (passive)} \\ \underline{\underline{A}}(\underline{q}_a, \underline{q}_d)\,\dot{\underline{q}}_d + \underline{\underline{B}}(\underline{q}_a, \underline{q}_d)\,\dot{\underline{q}}_a = \emptyset \end{array}\right]$ → $\underline{\Upsilon} + \underline{\underline{B}}^T\underline{\lambda} = \underline{\Upsilon}_a = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}_a}\right)^T - \left(\frac{\partial L}{\partial \underline{q}_a}\right)^T$

$\underline{\underline{A}}^T\underbrace{\underline{\lambda}}_{(\lambda\text{ LAGRANGE MULTIPLIERS})} = \underline{\Upsilon}_d = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}_d}\right)^T - \left(\frac{\partial L}{\partial \underline{q}_d}\right)^T$

$\underline{\underline{J}} = -\underline{\underline{A}}^{-1}\underline{\underline{B}}$ → $\left[\underline{\lambda} = \underline{\underline{A}}^{-T}\underline{\Upsilon}_d \; ; \; \underline{\Upsilon} = \underline{\Upsilon}_a + \underline{\underline{J}}^T\underline{\Upsilon}_d\right]$

② Body Dyn Principles - N.E. Law

[DERIVATIVE RULE : $\frac{d}{dt}(\vec{u})\Big|_{F_\emptyset} = \frac{d}{dt}(\vec{u})\Big|_{F_i} + \underline{\omega}_i \times \vec{u}$] $\sum \underline{f}_J = \frac{d}{dt}\Big|_{F_\emptyset}\underline{P}_J$; $\sum \underline{m}_J = \frac{d}{dt}\Big|_{F_\emptyset}\underline{h}_J(S_J)$

[FOR A BODY $\mathcal{B}_J$] → $\left[\begin{array}{l} \underline{P}_J = m_J\,\underbrace{\underline{V}_{S_J}}_{\text{[VELOCITY OF COM]}} \\ \underline{h}_J(S_J) = \underline{\underline{I}}_{S_J}\underline{\omega}_J \end{array}\right]$ [INERTIA MAT. EXPRESSED AT ITS COM]

↳ $\sum \underline{f}_J = (m_J\,\dot{\underline{V}}_J) + (\dot{\underline{\omega}}_J \times \underline{ms}_J) + (\underline{\omega}_J \times (\underline{\omega}_J \times \underline{ms}_J))$; $\sum \underline{m}_J = (\underline{\underline{I}}_{O_J}\dot{\underline{\omega}}_J) + (\underline{ms}_J \times \dot{\underline{V}}_J) + (\underline{\omega}_J \times (\underline{\underline{I}}_{O_J}\underline{\omega}_J))$

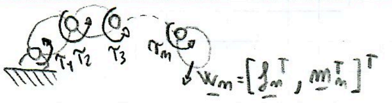
$\underline{W}_{t_J} = \begin{bmatrix} \sum \underline{f}_J \\ \sum \underline{m}_J \end{bmatrix} = \underline{\underline{M}}_J\,\dot{\underline{t}}_J + \underline{C}_J$

② Body Dyn Principles - Prin. Virtual Powers

[EQUIVALENT TO N.E. LAW!] ↑ LIKE IN CHAP ①

$\underline{f}_{O_J} + \underline{f}_{BJ} + m_J\underline{g} = m_J\dot{\underline{V}}_J + \underline{\omega}_J \times (\underline{\omega}_J \times \underline{ms}_J) + \dot{\underline{\omega}}_J \times \underline{ms}_J$ / $\underline{m}_{O_J} + \underline{m}_{BJ} + \underline{r}_{O_J B_J} \times \underline{f}_{BJ} + \hat{\underline{ms}}_J\,\underline{g} = \underline{ms}_J \times \dot{\underline{V}}_J + \underline{\omega}_J \times (\underline{\underline{I}}_{O_J}\underline{\omega}_J) + \underline{\underline{I}}_{O_J}\dot{\underline{\omega}}_J$

INPUT/OUTPUT:



$\underline{\tau} = [\tau_1, \dots, \tau_m]^T = \underline{J}_m^T \underline{W}_m$ WITH $\underline{J}_n$ S.T. ${}^{\phi}\underline{t}_m = \underline{J}_m \dot{\underline{q}}$ (TWIST OF END EFFECTOR)

③ LAGRANGE FORMALISM

FOR BODY i: $V_i = \frac{d}{dt}\Big|_{F_\phi} \overrightarrow{O_\phi O_i}$; $\omega_i = \omega_{i-1/\phi} + \omega_{i/i-1} = {}^i\omega_{i-1} + \dot{q}_i \underline{z_i}$

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)^T - \left(\frac{\partial L}{\partial q}\right)^T = \underbrace{\underline{M}_t(\underline{q}_t)}_{\text{INERTIA MATRIX OF THE ROBOT}} \cdot \ddot{\underline{q}}_t + \underbrace{\underline{\underline{C}}(\underline{q}_t, \dot{\underline{q}}_t)}_{\text{VECTOR OF CORIOLIS AN CENTRIFUGAL TORQUES}} \dot{\underline{q}}_t + \underbrace{\underline{f}_g(\underline{q}_t)}_{\text{VECTOR OF GRAVITY GENERALIZED EFFORTS}} \rightarrow \left[\underline{C}\, \dot{\underline{q}}_t = \dot{\underline{M}}_t \dot{\underline{q}}_t - \frac{\partial E}{\partial \underline{q}_t}\right] \& \left[f_{gi} = \partial V / \partial q_i\right]$$

($\underline{q}_t \to$ "total")

$C_{ij} = \sum_{k=1}^{n} c_{i,jk}\, \dot{q}_k$

$c_{i,jk} = \frac{1}{2}\left[\frac{\partial M_{ij}}{\partial q_k} + \frac{\partial M_{ik}}{\partial q_j} + \frac{\partial M_{jk}}{\partial q_i}\right]$

CONSIDERING INERTIA ACTUATORS:

KINETIC ENERGY OF ACTUATOR + GEARBOX: $E_{ACT_i} = \frac{1}{2} I_{a_i} \dot{q}_i^2$

(FOR PRISMATIC JOINT, I_{a_i} IS AN EQUIVALENT MASS) $\hookrightarrow I_{a_i} = \underbrace{N_i^2}_{\text{TRANSMISSION RATIO}} \cdot \underbrace{I_{m_i}}_{\text{INERTIA OF ROTOR AN TRANSMISSION BOF ACTUATOR } i}$

$$\hookrightarrow \left[\underline{\tau} = \underline{M}_t(\underline{q}_t)\ddot{\underline{q}}_t + \underline{I}_a \ddot{\underline{q}}_t + \underline{\underline{C}}(\underline{q}_t, \dot{\underline{q}}_t)\dot{\underline{q}}_t + \underline{f}_g(\underline{q}_t) + \underbrace{\underline{\underline{F}}_v \dot{\underline{q}}_t + \underline{f}_s}_{\text{IF YOU CONSIDER ALSO FRICTION}}\right]$$

$\hookrightarrow (= diag(I_{a_1} \cdots I_{a_m}))$

$\tau_{f_i} = f_{s_i} \cdot sign(\dot{q}_i) + f_{v_i} \dot{q}_i$

$\hookrightarrow \underline{F}_v = diag(f_{v_1} \cdots f_{v_m})$

$\hookrightarrow \underline{f}_s = [f_{s_1} sign(\dot{q}_1) \cdots]$

[FULL EQUATION (ACTUATORS + FRICTION)]! IDM $\to$ FOR THE DDM

$$\ddot{\underline{q}}_t = (\underline{M}_t + \underline{I}_a)^{-1}\left[\tau - \underline{\underline{C}}\,\dot{\underline{q}}_t - \underline{\underline{F}}_v \dot{\underline{q}}_t - \underline{f}_s - \underline{f}_g\right]$$

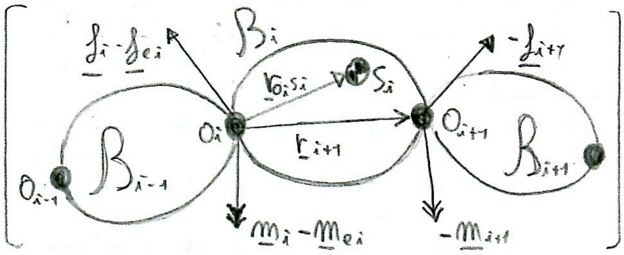
④ N-E FORMALISM

$\boxed{\text{FOR ROTATIONAL: } \overrightarrow{O_{i-1}O_i} = l_{O_{i-1}O_i} \cdot \underline{x_{i-1}} \quad \text{FOR PRISMATIC: } (l_{O_{i-1}O_i} + q_i)\,\underline{x_{i-1}}}$!

FOR BODY i: $V_i = \frac{d}{dt}\Big|_{F_\phi} \overrightarrow{O_\phi O_i}$; $\dot{V}_i = \frac{d}{dt}\Big|_{F_\phi} V_i$; $\omega_i = \omega_{i-1/\phi} + \omega_{i/i-1} = {}^i\omega_{i-1} + \dot{q}_i \underline{z_i}$; $\dot{\omega}_i = \frac{d}{dt}\Big|_{F_\phi} \omega_i$

FORWARD RECURSIVE EQUATIONS:

INITIAL CONDITIONS: ${}^\phi\underline{\omega}_\phi = \phi$; ${}^\phi\dot{\underline{\omega}}_\phi = \phi$; ${}^\phi\dot{\underline{V}}_\phi = -{}^\phi\underline{g}$



${}^i\underline{\omega}_{i-1} = {}^i\underline{R}_{i-1}\, {}^{i-1}\underline{\omega}_{i-1}$

${}^i\underline{\omega}_i = {}^i\underline{\omega}_{i-1} + \bar{\sigma}_i\, {}^i\underline{a}_i\, \dot{q}_i$

${}^i\dot{\underline{\omega}}_i = {}^i\underline{R}_{i-1}\, {}^{i-1}\dot{\underline{\omega}}_{i-1} + \bar{\sigma}_i({}^i\underline{a}_i \ddot{q}_i + {}^i\underline{\omega}_{i-1} \times {}^i\underline{a}_i \dot{q}_i)$

$({}^i\underline{V}_i = {}^i\underline{V}_{i-1} + {}^i\underline{\omega}_{i-1} \times ({}^i\underline{R}_{i-1}\, {}^{i-1}\underline{r}_{O_{i-1}O_i}) + \sigma_i\, {}^i\underline{a}_i \dot{q}_i)$ ← (NOT NEEDED)

${}^i\dot{\underline{V}}_i = {}^i\underline{R}_{i-1}\, {}^{i-1}\dot{\underline{V}}_{i-1} + {}^i\underline{R}_{i-1}\, {}^{i-1}\underline{U}_{i-1}({}^{i-1}\underline{r}_i) + \sigma_i({}^i\underline{a}_i \ddot{q}_i + 2\,{}^i\underline{\omega}_{i-1} \times {}^i\underline{a}_i \dot{q}_i)$

${}^i\underline{U}_i = {}^i\hat{\dot{\omega}}_i + {}^i\hat{\omega}_i\, {}^i\hat{\omega}_i$

${}^i\Sigma\underline{f}_i = m_i\, {}^i\dot{\underline{V}}_i + {}^i\underline{U}_i\, {}^i\underline{mS}_i$

${}^i\Sigma\underline{m}_i = {}^i\underline{I}_{O_i}\, {}^i\dot{\underline{\omega}}_i + {}^i\underline{mS}_i \times {}^i\dot{\underline{V}}_i + {}^i\underline{\omega}_i \times ({}^i\underline{I}_{O_i}\, {}^i\underline{\omega}_i)$

$\boxed{{}^{i-1}\underline{r}_i = {}^i\underline{\underline{R}}_{i-1}\, {}^{i-1}r_{O_{i-1}O_i}}$

$\boxed{{}^i\underline{a}_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

${}^i\hat{\omega}_i = \begin{bmatrix} \phi & -\omega_z & +\omega_y \\ +\omega_z & \phi & -\omega_x \\ -\omega_y & \omega_x & \phi \end{bmatrix}$

EXAMPLES OF $\underline{f}_{ei}$: $(m_i \underline{g})$ [WEIGHT OF $\mathcal{B}_i$]

EXAMPLES OF $\underline{m}_{ei}$: $(\underline{mS}_i \times \underline{g})$ [MOMENT DUE TO WEIGHT]

$(\overrightarrow{O_iO_{i+1}}) \times (-\underline{f}_{i+1})$ [MOMENT DUE TO $-\underline{f}_{i+1}$ EFFECT]

$(-\tau_{i+1})$ [ACTUATOR TORQUE OF JOINT $i+1$ ON $\mathcal{B}_i$]

(INSIDE $-\underline{m}_{i+1}$???)

BACKWARD RECURSIVE EQUATIONS:

${}^i\underline{f}_i = {}^i\Sigma\underline{f}_i + {}^i\underline{f}_{i+1} + {}^i\underline{f}_{ei}$

${}^{i-1}\underline{f}_i = {}^{i-1}\underline{R}_i\, {}^i\underline{f}_i$

${}^i\underline{m}_i = {}^i\Sigma\underline{m}_i + ({}^i\underline{R}_{i+1}\, {}^{i+1}\underline{m}_{i+1} + {}^i\underline{r}_{i+1} \times {}^i\underline{f}_{i+1}) + {}^i\underline{m}_{ei}$

$\tau_{t_i} = (\sigma_i\, {}^i\underline{f}_i + \bar{\sigma}_i\, {}^i\underline{m}_i)^T\, {}^i\underline{a}_i + \underbrace{I_{a_i}\ddot{q}_i + \tau_{f_i}}_{\text{(IF I ADD ROTOR INERTIA AND FRICTION)}}$

IF ACTUATED REVOLUTE: $\underbrace{\underline{f}_i = \begin{bmatrix} f_{ix} \\ f_{iy} \\ f_{iz} \end{bmatrix}}_{\text{LINEAR}}$; $\underbrace{\underline{m}_i = \begin{bmatrix} m_{ix} \\ m_{iy} \\ \phi \end{bmatrix} ; \underline{\tau}_i \begin{bmatrix} \phi \\ \phi \\ \tau_i \end{bmatrix}}_{\text{ANGULAR}}$

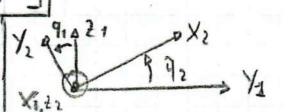
IF ACTUATED PRISMATIC: $\underbrace{\underline{f}_i = \begin{bmatrix} f_{ix} \\ f_{iy} \\ \phi \end{bmatrix} ; \underline{\tau}_i = \begin{bmatrix} \phi \\ \phi \\ \tau_i \end{bmatrix}}_{\text{LINEAR}}$; $\underbrace{\underline{m}_i = \begin{bmatrix} m_{ix} \\ m_{iy} \\ m_{iz} \end{bmatrix}}_{\text{ANGULAR}}$

⑦ DYN. MOTION CONTROL

FOR TWISTS:	FOR WRENCHES:
$\begin{bmatrix} {}^jV_j \\ {}^j\omega_j \end{bmatrix} = {}^j\underline{T}_i \begin{bmatrix} {}^iV_i \\ {}^i\omega_i \end{bmatrix}$	$\begin{bmatrix} {}^jf_j \\ {}^jm_j \end{bmatrix} = {}^j\underline{T}_i^{-T} \begin{bmatrix} {}^if_i \\ {}^im_i \end{bmatrix}$

${}^j\underline{T}_i = \begin{bmatrix} {}^jR_i & -{}^jR_i\, {}^i\hat{r}_j \\ \phi_3 & {}^jR_i \end{bmatrix}$ ${}^j\underline{T}_i^{-1} = \begin{bmatrix} {}^iR_j & {}^i\hat{r}_j\, {}^iR_j \\ \phi_3 & {}^iR_j \end{bmatrix}$

$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$



$${}^2\underline{R}_1: \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} = \begin{array}{c} \begin{matrix} {}^2x_1 & {}^2y_1 & {}^2z_1 \end{matrix} \\ \begin{bmatrix} \phi & c_2 & s_2 \\ \phi & -s_2 & c_2 \\ 1 & \phi & \phi \end{bmatrix} \end{array} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix}$$

$(a \times b) \cdot c = (b \times c) \cdot a = (c \times a) \cdot b$

$\frac{d}{dt}\Big|_{F_\phi}(\vec{u}) = \frac{d}{dt}\Big|_{F_i}(\vec{u}) + (\underline{\omega}_i \times \vec{u})$