Mechanics of Mechanisms and Machines
&
Advanced Modelling and Simulation Techniques for Robots

Lecture Notes

UniGe

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○ Mechanics of Mechanisms and Machines:

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References Not duly following:

- Hunt, K., 1978, *Kinematic geometry of mechanisms*, Clarendon Press.
- Murray, R.M, Li, Z., and Sastry, S.S., 1994, *Mathematical introduction to robotic manipulation*, CRC.
- John Joseph Uicker, J.J., G. R. Pennock, G.R., and Shigley, J.E., 2016, *Theory of Machines and Mechanisms*. 5th ed. New York: Oxford University Press.

Contents Fundamentals of theory of mechanisms and machines: synthesis, analysis, modelling, singularities. Kinematics and elements of dynamics. Serial and parallel architectures. Compliant mechanisms. Architectures for robotics. The Lie group of rigid body displacement. Screw theory.

1. Linear spaces, screws, twists, and wrenches: the basics of screw theory.
2. Application: constraint analysis and synthesis of parallel manipulators.
3. Kinematic geometry of planar mechanisms.
4. Velocity and singularity analysis.
5. Statics of mechanisms.
6. Acceleration in rigid-body systems, introduction to dynamics.

The course provides the fundamentals of kinematic geometry. On this basis, the students will be ready to further improve their skills and knowledge and be able to handle various advanced problems arising in the mechanics of robotic systems. In particular, they will have the proper mathematical-modelling foundation to attain more specialized skills and knowledge in areas such as multi-body dynamics or flexibility analysis, which are often of crucial importance in robotics-engineering applications.
An important emphasis of the course is on correcting and developing students? geometrical intuitions for rigid-body motion in three-dimensional space. For this purpose, visualizations and classical geometry are used in parallel with rigorous mathematical formalisms.

○ Advanced Modelling and Simulation Techniques for Robots:

Contact Dimiter Zlatanov (zlatanov@dimec.unige.it)

Contents Give the students the fundamentals of:

- C++ programming
- Industrial robot manipulator programming using specialized robot languages.

Contents

1 Introduction

1.1 $\mathbb{E}^3$ & Rigid Bodies

An **affine euclidean space** $\mathbb{E}^3$ is "our normal space". $\mathbb{E}^3 = \mathbb{R}^3/\mathbf{0}$ so there is no "special point" as the origin, and the elements are **points**, not vectors (it's a fine space, not a vector space). We never use $\mathbb{R}^3$ in robotics!

Which point we will consider "inside" the robot? If the robot it's still to be designed, it is impossible to say. So, at a first approximation we don't know → when we use the term "**rigid body**" we don't refer to something with a certain shape: there is **no shape** in rigid body.

A rigid body **displacement** in $\mathbb{E}^3$ preserves distance & orientation. Beware though! Orientation is not our common-sense "orientation", but the left-handed/right-handed orientation: it does not switch left and right.

"Mechanical Philosophy" approach: the only things to be considered are rigid particles in 3D space acting through direct contact (→ "the Universe is a machine" – The 17th c. Scientific Revolution: Galileo, Leibniz, Descartes). But is this really true? Not really:

- Newton's Law of Gravitation: bodies act at a distance: **nothing** is a machine
- Einstein's General Relativity: Space is curved: Real geometry is **not** Euclidean

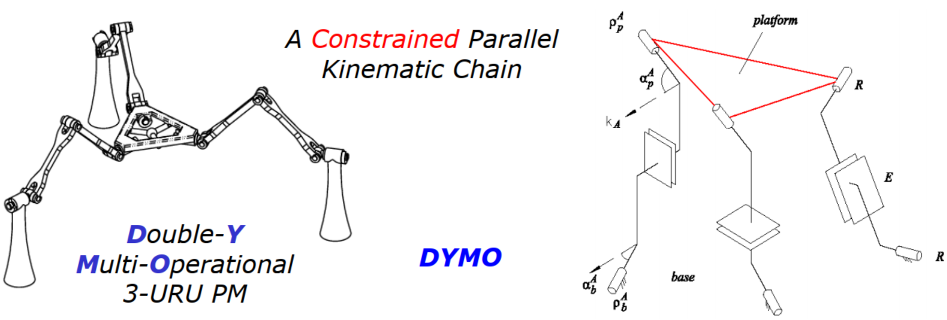
So, it is only an approximation (**idealization**):

- **Rigid body**: only a special case of a continuum
- $\mathbb{E}^3$: only a special manifold

So, rigid bodies are **unreal but indispensable**, because the approximation is good enough and simplification is important (rigid bodies and Euclidean spaces are simpler than curved continua). Moreover, machines exists **in our minds**:

- **Rigid body**: are our inborn world model (Rigidity principle + body permanence)
- $\mathbb{E}^3$: is our intuition about space

A rigid body has 6 DoF normally, but for example in parallel robots they are less when connected in a chain (motion constraints). A 3-legs parallel robot with 3 DoF can have for example 3 translational DoF and no rotational capabilities, or 3 rotational DoF and no translational capabilities, and so on...



- Translational mode → 3D translation
- Rotational mode → 3D rotation
- Planar mode → 2D translation 1D rotation

Sometimes we can design robots that can switch from one mode to another (planar-to-translational modes etc...) depending on the **active constraints** on the mechanism. We can have other special kind of motions, like in the lockup mode (0 DoF) and the CVC mode (Constant Velocty Shaft mode, normally used to transmit motion through shafts able to rotate arbitrarily while maintaining their angular velocity).

Real-life example: a 3-UPU translational robot designed and built un Korea had a malfunction: the platform was rotating even with actuators locked, even with the three prismatic joint lengths equal → we will be able to explain why this happens, and we will do it with screw theory.

1.2 Screw theory

Historically, in the beginning analytic tools were rudimentary (before Calculus, Newton, Euler and Lagrange) and mechanics was a **geometric** art. Then, with **Lagrange**, the "analytical revolution" brought clearness in the field. Analytical mechanics analyse mechanics without any reference to the figurative/-geometrical approach. This lead to a decrease of interest on the main bases of mechanical motion, that are "essentially pictorial, geometrical. They arise from natural philosophy. Students in the mechanical sciences are becoming increasingly unable to contemplate a piece of ordinary reality in machinery accordingly, and to extract from that reality the geometric essence of it" (– *Freedom in Machinery*, J. Phillips, 1984).

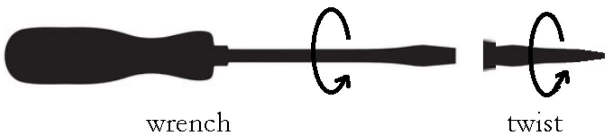
Nowadays modern computers are doing the "hard work", and people tend to know less. But **we need geometry** because it allows us to have a complex space intuition, while the computer will deal with the mechanical part of this complexity (**computer cannot design**).

So, why we have to use screw theory? Because it is a powerful geometrical tool (→ major aid to intuition) and it is relatively simple (because it deals with linear spaces) → it helps us to "see what's inside the computer".

Screw theory – Traditional approach Ball (1900) , Hunt (1978), Phillips (1984)

- A screw is a line with a pitch, like the spires of an actual screw along a line axis
- Along the screw a body can be subject to:
  a **twist** (rotation, translation)
  a **wrench** (force, moment)

Given, in a **twist**, the velocity of the "helical motion" at that instant t, you will fully know the **motion** of the rigid body in that instant t.



Screwdriver analogy: the twist represents the motion, the wrench "moves" the twist with a push&rotate action (screwdriver).

Traditional screw theory uses *traditional mathematical methods*:
– Little mathematical background
– Reveals the geometric nature of motion
.. and uses *classical line geometry*:
– A lot of geometric intuition
– Can intimidate the student

Screw theory – Modern approach Murray, Li, Sastry (1993) Selig (1996, 2005)

- A screw is an element of $se(3)$ group

6-Dimensional Lie Group	$\rightarrow$	$SE(3)$: displacements
The Lie Algebra related to $SE(3)$	$\rightarrow$	$se(3)$: twists
and its dual space	$\rightarrow$	$se(3)^*$: wrenches

Modern screw theory approach uses *complex mathematical methods*:
– Requires advanced math background
– Highly rigorous and abstract
... and uses *Group theory* and *differential geometry*:
– The language of the new mechanics
– Reveals the math essence of motion
– Can intimidate the student

Screw theory – Summer Screws Approach Merging the new and the old

- A screw is an element of a linear space

6-Dimensional space	$\rightarrow$	Twists
Its dual space	$\rightarrow$	Wrenches
Preferred bases	$\rightarrow$	Plücker coordinates for frames in 3D space
Invariance	$\rightarrow$	Change of frame
Linear subspaces	$\rightarrow$	Screw systems

The summer screw approach uses *vector algebra*:
– Minimize abstract mathematics
– Be able to calculate
... and provides *geometric emphasis*:
– Maximize insight
– Be able to see and imagine

1.3 Vector spaces

A **vector space** (or linear space) V over the field $\mathbb{R}$ is a set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}, ...\} \in V$ of vectors with 2 operations defined:

VA Vector addition: $V \times V \rightarrow V, \quad (\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{u} + \boldsymbol{v}$

Associative law:	$(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w} = \boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$
Commutative law:	$\boldsymbol{u}+\boldsymbol{v} = \boldsymbol{v}+\boldsymbol{u}$
Zero vector:	$\exists \boldsymbol{0} \;\; s.t. \;\; \forall \boldsymbol{u}, \;\; \boldsymbol{0}+\boldsymbol{u} = \boldsymbol{u}$
Opposite vector:	$\forall \boldsymbol{u} \exists (-\boldsymbol{u}) \;\; s.t. \; (-\boldsymbol{u})+\boldsymbol{u} = \boldsymbol{0}$

SM Scalar multiplication: $\mathbb{R} \times V \rightarrow V, \quad (\lambda, \boldsymbol{v}) \mapsto \lambda \boldsymbol{u}$

Associative law:	$\lambda(\mu \boldsymbol{u}) = (\lambda\mu)\boldsymbol{u}$
Distributive law:	$(\lambda+\mu)\boldsymbol{u} = \lambda\boldsymbol{u}+\mu\boldsymbol{u}$
Distributive law:	$\lambda(\boldsymbol{u}+\boldsymbol{v}) = \lambda\boldsymbol{u}+\lambda\boldsymbol{v}$
Unit scalar:	$1\boldsymbol{u} = \boldsymbol{u}$

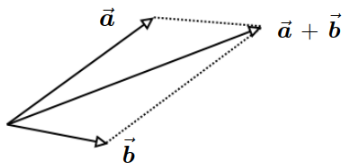
Basic vector spaces examples Let's see some basic vector spaces, some of them will be useful in the following pages:

- for $\emptyset \rightarrow$ **VA** and **SM** are trivially defined (it's the empty set)
- for $\{\boldsymbol{0}\} \rightarrow$ **VA** and **SM** are again trivially defined (you can check the properties)
- for number sets like $\mathbb{R}$, $\mathbb{Q}$,$\mathbb{C}$, $\mathbb{R}-\mathbb{Q}$ and for functions $\rightarrow$ **VA** and **SM** are again easily defined
- for ***n*-tuples** like $(x_1, ..., x_n) \in \mathbb{R}^n \rightarrow$ **VA** and **SM** are defined as acting component-wisely:

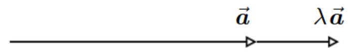
  $VA: \quad (x_1, ..., x_n) + (y_1, ..., y_n) = (x_1+y_1, ..., x_n+y_n)$
  $SM: \quad \lambda(x_1, ..., x_n) = (\lambda x_1, ..., \lambda x_n)$

- for "**arrows**" from a point in space (magnitude + direction):

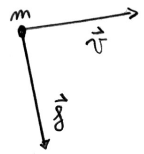
  VA $\rightarrow$ parallelogram rule:



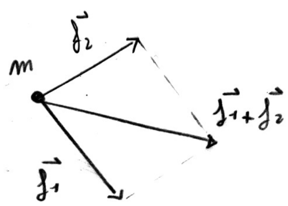
  SM $\rightarrow$ length dilatation:



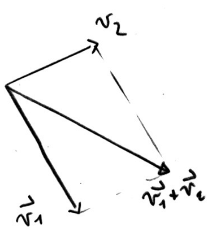
Let's dive in this last example w.r.t. forces and velocities. Let's consider a particle with mass m with a certain velocity and a certain force applied on it. In general physics we represent forces like this:



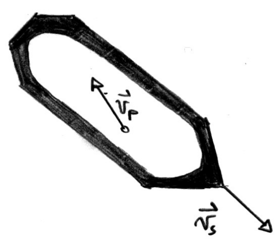
Let's consider only forces for now. If we have multiple forces, the resultant force is **equivalent** to the other two:



How does it work for velocities? The parallelogram rule as well it is allowed:



But this is not as straightforward as for the forces case, because we have a velocity at a time! So, imagine a ship carrying a container that moves with a certain velocity $\vec{v_p}$:



If I'm on the shore, the velocity of the container is the parallelogram-rule resultant velocity $v_p + v_s \rightarrow$ for chains of several joints, to compute the velocity of the End-Effector we will have to **compose** the joint **velocities**.

So, we can conclude that:

- For forces, we will have a **vector field** F^3 of the forces acting on a particle (arrow from the particle, magnitude + direction), for which:

  VA $\rightarrow$ resultant force (parallelogram rule)

  SM $\rightarrow$ proportional change of force intensity

$\Rightarrow$ So, a **force field** $\boldsymbol{f}(P)$ will map $\{\boldsymbol{f} | \boldsymbol{f} : E^3 \rightarrow F^3\}$ an arrow $\in F^3$ to every point $P \in E^3$

- Identically, for velocities, we will have M^3 as a **vector field** for the velocities on the particle (VA conceptually slightly different as seen before).

$\Rightarrow$ So, a **velocity field** $\boldsymbol{v}(P)$ will map $\{\boldsymbol{v} | \boldsymbol{v} : E^3 \rightarrow M^3\}$ an arrow $\in M^3$ to every point $P \in E^3$

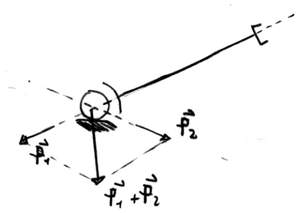
1.4 Forces and velocities vectors

Let's now make some advanced examples and counterexamples regarding forces-like (statics) and velocity-like (instantaneous kinematics) situations, in order to explain the relations contained in the following Table:

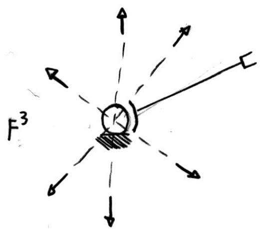
Statics		Instantaneous kinematics
FORCE $\mathbf{F}$ along a LINE of action	$\leftrightarrow$ both are LINE VECTORS	$\begin{cases}\text{INFINITESIMAL ROTATION} \\ \text{or} \\ \text{ANGULAR VELOCITY } \boldsymbol{\omega}\end{cases}$ about an AXIS of rotation
COUPLE $\mathbf{C}$	$\leftrightarrow$ both are FREE VECTORS	$\begin{cases}\text{INFINITESIMAL TRANSLATION} \\ \text{or} \\ \text{TRANSLATIONAL VELOCITY } \boldsymbol{\tau}\end{cases}$
WRENCH of INTENSITY F on a SCREW $\mathcal{S}$ (couple) : (force) $=$	$\leftrightarrow$ $=$ PITCH h $=$ of dimension [LENGTH]	INFINITESIMAL TWIST of AMPLITUDE ω about an ISA $\mathcal{S}$ $= \begin{pmatrix}\text{translational} \\ \text{velocity}\end{pmatrix} : \begin{pmatrix}\text{angular} \\ \text{velocity}\end{pmatrix}$
PURE FORCE	when PITCH $= 0$ $\leftrightarrow$	PURE ROTATION
PURE COUPLE	when PITCH $= \infty$ $\leftrightarrow$	PURE TRANSLATION
wrenches act 'IN PARALLEL' with one another		twists act 'IN SERIES' with one another

What does it mean to that "wrenches act **in parallel**"? It's what we have seen before with forces: you can have two or more forces acting at the same time (in parallel). Then, what about "twists act **in series**"? It's what we have seen before with forces: you can have only one v at each time, then you add them sequentially, in series (but no matter the order due to the commutativity of the sum).

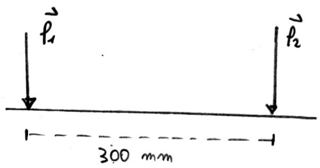
Ex.1) Forces that lead to forces Let's consider two forces $\vec{\varphi_1}$ and $\vec{\varphi_2}$ transmitting through a **spherical joint**:



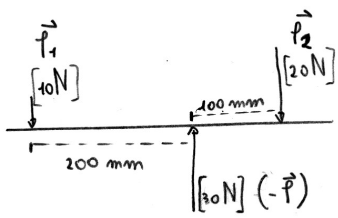
All the forces applied to the spherical joint will transmit through a single point → instead of $\vec{\varphi_1}$ and $\vec{\varphi_2}$ we can express $\vec{\varphi_1} + \vec{\varphi_2}$ and no physical experiment will tell us the difference (under **no** friction and **rigid body** hypothesis). So, **VA** and **SM** here hold, in fact we can clearly see the **vector space** F^3 as composed by all the forces that can be applied to the rigid body (e.g. $\vec{\varphi_1}, \vec{\varphi_2} \in F^3$):



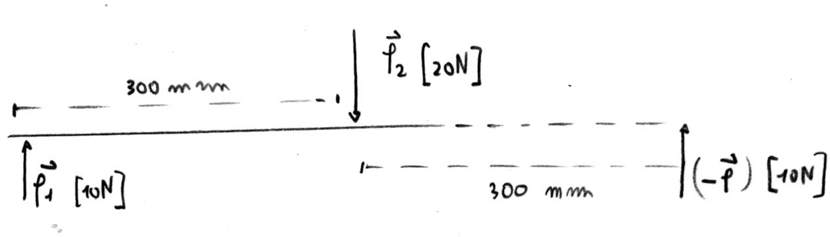
Ex.2) Forces that lead to forces and couples Let's consider now two forces $\vec{\varphi_1}$ and $\vec{\varphi_2}$ acting on a **lever**:



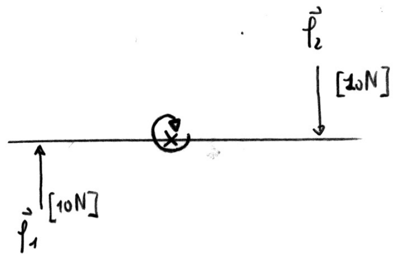
The forces apply on the same plane and in the same direction. S, in this case, what is $\vec{\varphi_1} + \vec{\varphi_2}$? Let's make a numerical example:



Here, we have $\vec{\varphi_1} + \vec{\varphi_2} + (-\vec{\varphi}) = 0$. The resultant of the sum of $(-\vec{\varphi})$ and the other two forces is the **equilibrium**. So, how do we balance the lever? Where we will apply the force? We can put of $(-\vec{\varphi})$ as already seen, but here there is **no parallelogram rule** defined! Let's make another example:



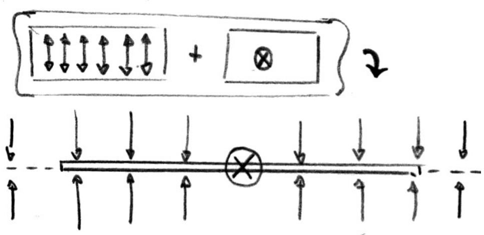
But this is **not correct**! The resultant does not take in account the rotation of the lever that we can intuitively imagine. So, let's consider the following example:



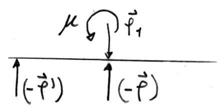
Here $(-\vec{\varphi})$ should be $0N$, but the lever is not still, it rotates! It is not equivalent to $\vec{\varphi_1}$ and $\vec{\varphi_2}$, because with an easy experiment this time we would be able to identify the difference from the two situations. When you have a **force couple** like this one, it will provide a **pure moment** $(\boldsymbol{r} \times \boldsymbol{E})$. In this case:

$$10[N] \cdot 300[mm] = 10[N] \cdot 0.3[m] = 3[N \cdot m]$$

If we allow force couples, we will have the following vector set composed by all the possible forces applicable to the lever and the one rotation axis $\perp$ to the plane where these forces lie:



Is this a vector space? We had before $\sum_i \vec{\varphi}_i + (-\vec{\varphi}) = 0$. Adding the pure moment mu, we will have $\vec{\varphi} + (-\vec{\varphi}) + \mu = 0$ and we will have again correct consequences, like the one in the following example (one force + one moment applied to the lever):

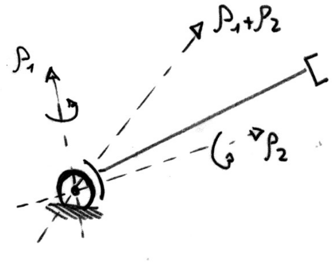


So, is this, after these considerations, this is a **space composed by all the parallel forces and by the couple along that axis.**

1+2) Forces & couples acting on a rigid body Let's summarize the evidences obtained in examples 1 and 2 w.r.t. rigid bodies. Two forces never has necessarily a force as a resultant, because they could lead also to pure moments (couples)

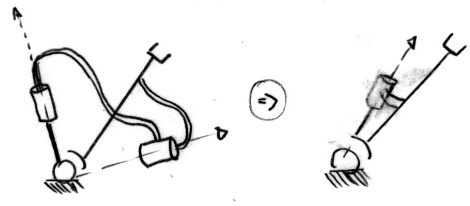
We found two kinds of vector spaces, the one composed by all the forces whose line of action pass through a point and the one composed by all the forces along parallel lines plus the couple along an axis. This double characterisation will be very useful after while talking of screw spaces.

Ex.3) Inst. rotations that lead to inst. rotations Let's consider two instantaneous rotations with the related angular velocities $\vec{\omega_1}$ and $\vec{\omega_2}$ on a **spherical joint**:



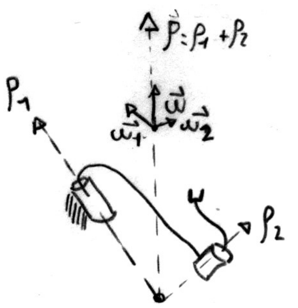
What is an instantaneous motion? It is like a motion but linearised: we can think about it as a very small motion, or a function changing with time approximated at Taylor's first order.

To think about it in physical terms, we can think about two R-joints. The sum will be the replacement of them with a R-joint along the parallelogram-rule resultant axis:



In fact, if we only have rotations whose axis pass through a single point, we can apply the parallelogram rule:

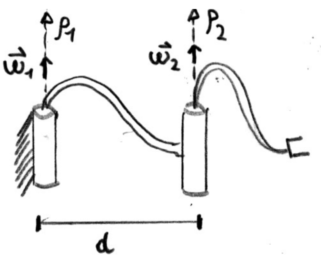
- **VA**: the resultant rotation, obtained applying the parallelogram rule on the rotation axis when the body is the end-effector of a RR chain with intersecting axes:

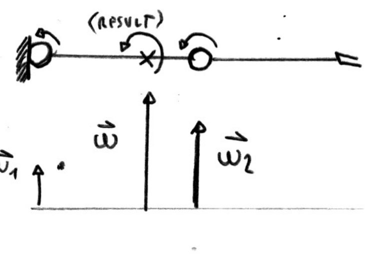


- **SM**: again the proportional change of rotational amplitude

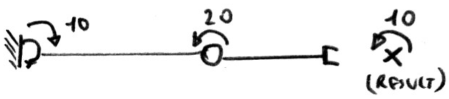
So, it is a **vector space** containing all the angular velocities $\vec{\omega_i}$ whose relative rotational axis pass through the single point of the spherical joint.

Ex.4) Inst. rotations that lead to inst. rotations & translations Let's consider two instantaneous rotations with the related angular velocities $\vec{\omega_1}$ and $\vec{\omega_2}$ whose axis are parallel (**planar RR** end-effector):

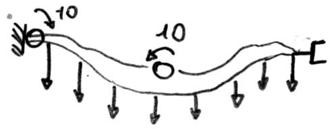




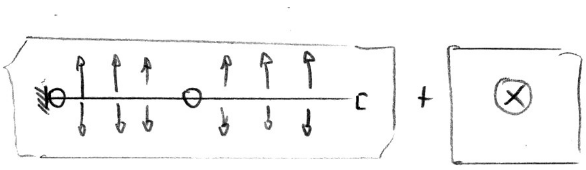
Let's say $\omega_1 = 10\ [s^{-1}]$ and $\omega_2 = 20\ [s^{-1}]$ with $d = 300\ [mm]$. In analogy with **Ex.2)**, we can fix the resultant as shown, without further computations: the resultant of the 2 instantaneous velocities like these ones (planar case) is another velocity in the plane, with the same direction. Consider also another example:



Here the resultant is found as in the lever case as well. But let's consider the "couples-analogy" example. Let's apply two opposite equal rotations:



In this case the resultant movement is a translation! We can intuitively notice it even better if we imagine an additional rigid extension as in the figure: all the points will have the same velocities. So, in analogy with **Ex.2)**, we will have all the rotations allowed only along one axis and a set of parallel vectors as possible velocities:



1+2+3+4) Conclusion There are two kinds of spaces, the "**radial**" ones and the "**parallel plus ⊥axis**" ones. A system of forces is not necessary similar to a result force, and some motions not always a resultant rotation/translation.

- Two forces have a "resultant force or couple" only if their axis are coplanar
- Two rotations have a "resultant rotation or translation" only if their axis are coplanar

2 Screw theory

2.1 Wrenches, twists, and screws

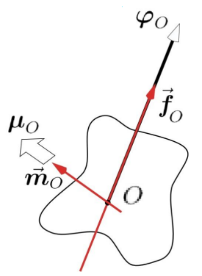
Consider an **external action** on a rigid body, on a specified continuous of particles B:

$$\{\vec{f}_P \mid P \in B\}$$

It is NOT necessary to specify on which particle a force is applied because this external actions (force fields) can be characterised in a simple way. Picking a point O in the body we can then represent the external action with two three-dimensional vectors:

- a **force** $\boldsymbol{\varphi}_O$ applied through this point (with direction and magnitude $\vec{f}_O$)
- a **couple** $\boldsymbol{\mu}_O$ (with moment $\vec{m}_O$)

$\Rightarrow$ we have a **system of forces** $\boldsymbol{\Phi}_O$ at O acting on a rigid body: $\boldsymbol{\Phi}_O = \{\boldsymbol{\varphi}_O, \boldsymbol{\mu}_O\}$



To add: apply "**in parallel**" (see later)

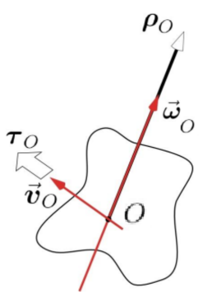
Consider now an **instantaneous motion** of a rigid body B (a specified continuum of particles):

$$\{\vec{v}_P \mid P \in B\}$$

It is again NOT necessary to specify every particle's velocity. In fact, if a rigid continuum moves, you don't have to specify for each particle the single velocity (this happens for a river, not for a rigid body). So, we can again pick a point O and the instantaneous motion will be described by two three-dimensional vectors:

- an **instant. rotation** $\boldsymbol{\rho}_O$ applied through this point (with direction and magnitude $\vec{\omega}_O$)
- an **instant. translation** $\boldsymbol{\tau}_O$ (with velocity $\vec{v}_O$)

$\Rightarrow$ we have an **instant. motion** $\boldsymbol{\Upsilon}_O$ at O of a rigid body: $\boldsymbol{\Upsilon}_O = \{\boldsymbol{\rho}_O, \boldsymbol{\tau}_O\}$



To add: apply "**in series**" (see later)

> After what we saw, we know that for rigid bodies, we have to specify just one force and one couple, and from them you can get the description for every point. What about motions? Normally, you have to specify velocity fields as well. For rigid bodies those velocity fields can be obtained very easily with a rotation and a translation, so that every motion can be realized by RP mechanisms. But now the question is: which force/couple and rot/transl couple we want to specify?

Let's consider the following case regarding **statics**. We want to move our previous point from O to another point O'. The **shifting law** gives you the motion from a point to another. If you have a force and also a moment, and you want to move to a new configuration $O \to O'$:

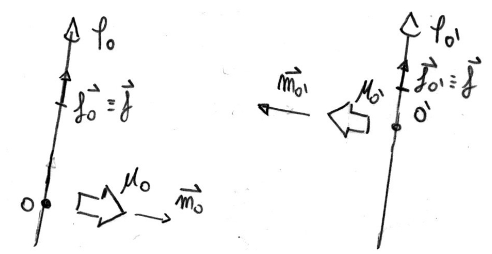
- The **total force** $\vec{f}$ has to be the same:

$$\vec{f_O} = \vec{f_{O'}} = \vec{f}$$

- The **moment** in the new configuration will be:

$$\vec{m_O} = \vec{m_{O'}} + \vec{OO'} \times \vec{f} \qquad \vec{m_{O'}} = \vec{m_O} + \vec{O'O} \times \vec{f} = \vec{m_O} + \vec{f} \times \vec{OO'}$$

Graphically:



So, in this case, under the shifting rule assumptions, the two systems $\mathbf{\Phi}_O$ and $\mathbf{\Phi}_{O'}$ are **equivalent**. We don't want to distinguish all the equivalent systems, but consider them all together $\to$ the **equivalence class** $\zeta = |\Phi|$ is called a **wrench**.

Thanks to the wrench, we don't have to distinguish the different but equivalent configurations given by the shifting law: the wrench is $(\vec{f}, \vec{m_O})$ at O and $(\vec{f}, \vec{m'_O})$ at O'.

Let's consider now this other case on **instantaneous kinematics**. We want to move, again, from a O configuration to another O'. The **shifting law** is:

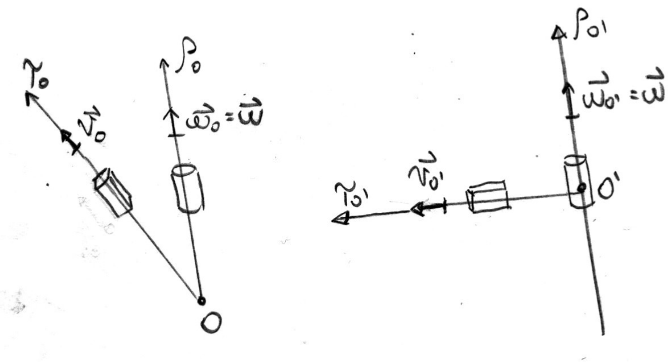
- The **angular velocity** $\vec{\omega}$ has to be the same:

$$\vec{\omega_O} = \vec{\omega_{O'}} = \vec{\omega}$$

- The **velocity** in the new configuration will be:

$$\vec{v_O} = \vec{v_{O'}} + \vec{OO'} \times \vec{f} \qquad \vec{v_{O'}} = \vec{v_O} + \vec{O'O} \times \vec{f} = \vec{v_O} + \vec{f} \times \vec{OO'}$$

Graphically:



So, again, under the shifting rule assumptions, the two systems $\boldsymbol{\Upsilon}_O$ and $\boldsymbol{\Upsilon}_{O'}$ are **equivalent**. Considering them all together → the **equivalence class** $\xi = |\Upsilon|$ is called a **twist**.

Thanks to the twist, we don't have to distinguish the different but equivalent configurations given by the shifting law: the wrench is $(\vec{\omega}, \vec{v_O})$ at O and $(\vec{\omega}, \vec{v'_O})$ at O'.

That is the reason why, if for $\vec{\omega}$ we can speak of "angular velocity", $\vec{v_O}$ is **not a "translational" velocity**! In fact, it is the velocity of a point that, at a certain time passes through the point O, and its velocity happens to be $\vec{v_O}$. $\vec{v_O}$ in fact is the **velocity at the origin** (origin not as a specific privileged point of the body but as all the points from where along the velocity direction passing through the configuration point O).

> So, to summarize:
>
> A **wrench** is a system of forces (reduced at a point) with equivalent systems identified.
> A wrench is an entity **invariant** of frame choice.
> For a given origin, O, it is given by a pair of vectors $\boldsymbol{\zeta} = (\vec{f}, \vec{m_O})$:
> – the resultant force
> – the moment at O
>
> A **twist** is an instantaneous motion (reduced at a point) with equivalent motions identified.
> A twist is an entity **invariant** of frame choice
> For a given origin, O, it is given by a pair of vectors $\boldsymbol{\xi} = (\vec{\omega}, \vec{v_O})$:
> – the body angular velocity
> – the velocity of the point coinciding with O

2.2 Wrench and twist spaces

Remembering our previous analysis of forces and velocities vector spaces:

- Two forces have a "resultant force or couple" only if their axis are coplanar
- Two rotations have a "resultant rotation or translation" only if their axis are coplanar

The word "coplanar" subsumes two seen situations, the case of incident lines (→ in series) and parallel lines (→ in parallel). From these configurations, we can easily see why these are the properties of wrench and twist spaces, mutated from the already seen vector spaces of forces and velocities:

- Wrenches form a vector space, $se^*(3)$:

  VA: **resultant wrench** → if you apply two forces to a rigid body (all forces acting at the same time, in **parallel**) the resultant force will be:

$$\text{at } O: \qquad \zeta + \zeta' = (\vec{f}, \vec{m_O}) + (\vec{f'}, \vec{m'_O}) = (\vec{f} + \vec{f'}, \vec{m_O} + \vec{m'_O})$$

SM: proportional increase of **intensity**:

$$\text{at } O: \qquad \lambda\zeta = \lambda(\vec{f}, \vec{m_O}) = (\lambda\vec{f}, \lambda\vec{m_O})$$

[VA and SM do not depend on O (show it)]

- Twists form a vector space, $se(3)$:

  VA: **resultant motion** $\rightarrow$ all motion acting in **series** (not at the same time, but in any time order! that is because consequentiality is an appearance, they are commutative):

  $$\text{at } O: \qquad \xi + \xi' = (\vec{\omega}, \vec{v_O}) + (\vec{\omega'}, \vec{v'_O}) = (\vec{\omega} + \vec{\omega'}, \vec{v_O} + \vec{v'_O})$$

  SM: proportional increase of **intensity**:

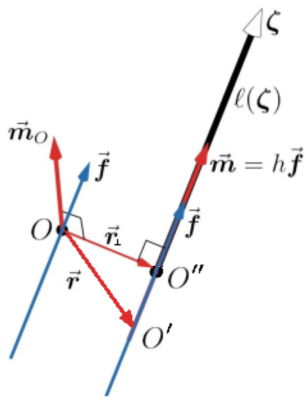
  $$\text{at } O: \qquad \lambda\xi = \lambda(\vec{\omega}, \vec{v_O}) = (\lambda\vec{\omega}, \lambda\vec{v_O})$$

  [VA and SM do not depend on O (show it)]

2.3 Canonical representation

It is possible to represent a wrench/twist in million ways. Are there some representations that are better than others? You must represent it at some point, but for some actions/motions in fact there is not a fixed point (e.g. for couples/translations). Nevertheless, there is a way to simplify: we can find a point in which the total force/angular velocity is parallel to the total moment/velocity at the origin.

So, starting from a **wrench** at O, we have to search for a point: P s.t. $\vec{m_P} /\!/ \vec{f}$



We have two cases:

- $\vec{f} = 0 \rightarrow \zeta = (\mathbf{0}, \vec{m_O})$, $\forall$ points of the rigid body
- $\vec{f} \neq 0 \rightarrow$ having $\vec{m_P} /\!/ \vec{f}$ means that $\exists!$ line $\ell(\zeta) = \{\vec{r} + \lambda\vec{f}/|\vec{f}|\}$ (the **screw axis**) such that:

  $$\vec{m}_P = h\vec{f}, \ \forall P \in \ell(\zeta)$$

So, at O thaks to the shifting law, we have $\vec{m}_P = h\vec{f} = \vec{m}_O + \vec{O'O} \times \vec{f}$. Dot-multiplying everything for $\vec{f}$ we obtain:

$$h\vec{f} \cdot \vec{f} = \vec{m}_O \cdot \vec{f} + \underbrace{(\vec{O'O} \times \vec{f}) \cdot \vec{f}}_{=0} = \vec{m}_O \cdot \vec{f}$$

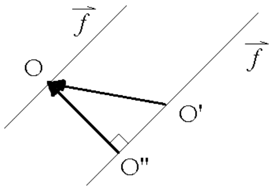
from which we can extract:

$$h = \frac{\vec{f} \cdot \vec{m_O}}{\vec{f} \cdot \vec{f}}$$

Then cross-multiplying instead everything for $\vec{f}$ we obtain:

$$\underbrace{\vec{f} \times h\vec{f}}_{=0} = \vec{f} \times \vec{m}_O + \vec{f} \times (\vec{O'O} \times \vec{f}) = 0$$

Let's consider then another point O'' as shown in the following Figure:



We know that $\vec{O'O} = \lambda\vec{f} + \vec{O''O}$ and that $\vec{O'O''}$ is proportional to $\lambda\vec{f}$, so:

$$\vec{O'O} = \vec{O'O''} + \vec{O''O} \quad \rightarrow \quad (\vec{O'O} \times \vec{f}) \times \vec{f} = 0 + (\vec{O''O} \times \vec{f}) \times \vec{f} = (\vec{f} \cdot \vec{f})\vec{O''O}$$

from which we can extract:

$$\vec{O''O} = \frac{\vec{f} \times \vec{m_O}}{\vec{f} \cdot \vec{f}}$$

The vector is $\vec{O''O}$ because it does not mind where is O' exactly on the line of direction of $\vec{f}$ and $\vec{m_O}$, the moment is the same on this line, so every point on the line has the same force and moment! We found our $\vec{r_\perp}$:

$$\vec{r_\perp} = \frac{\vec{f} \times \vec{m_O}}{\vec{f} \cdot \vec{f}}$$

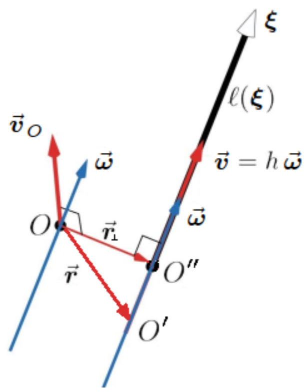
The **axis of the wrench** then – the line for which the force and the moment have the same direction, is given by:

$$\text{axis: } \{O'' + \lambda\vec{e} \mid \forall\lambda \in \mathbb{R}\} = \{\vec{r_\perp} + \lambda\vec{e} \mid \forall\lambda \in \mathbb{R}\} \quad (\vec{e} = \text{direction of } f)$$

Thus, the wrench at O will be expressed by:

$$\boxed{\zeta = (\vec{f}, \vec{m_O}) = (\vec{f}, h\vec{f} + \vec{r} \times \vec{f}) \qquad h = \frac{\vec{f} \cdot \vec{m_O}}{\vec{f} \cdot \vec{f}} \qquad \vec{r}_\perp = \frac{\vec{f} \times \vec{m_O}}{\vec{f} \cdot \vec{f}} \qquad \text{axis: } = \{\vec{r_\perp} + \lambda\vec{e} \mid \forall\lambda \in \mathbb{R}\}}$$

For the **twists** it is similar to what seen for the wrenches:



$$\boxed{\xi = (\vec{\omega}, \vec{v_O}) = (\vec{\omega}, h\vec{\omega} + \vec{r} \times \vec{\omega}) \qquad h = \frac{\vec{\omega} \cdot \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} \qquad \vec{r}_\perp = \frac{\vec{\omega} \times \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} \qquad \text{axis: } = \{\vec{r_\perp} + \lambda\vec{e} \mid \forall\lambda \in \mathbb{R}\}}$$

For any motion (except the $\vec{\omega} = 0$ pure translation case) there is a special line whose direction is the direction of $\vec{\omega}$. It is "special" because the body around this line moves with $\vec{\omega} = 0$ angular velocity (rotation) and translates parallel to $\vec{\omega}$. This motion is called **screwing**, because involves rotation and translation parallel to the rotation, and is characterised by an axis in which points have minimum velocity because they translate but don't rotate → it is called **screw axis**. The "step" of this helical movement is called **pitch** and it's the already seen parameter h.

A line ℓ with a pitch h (a metric quantity) is a geometric element called a **screw**. The screw of a couple/translation has no axis, only a direction → **infinite pitch screw** ($h = \infty$).

A (geometric) screw is **not a vector**. Screws form the **projective space** underlying the space of twists and wrenches (<u>note</u>: the projective space of V is obtained by identifying $\vec{v} \sim \lambda\vec{v}$).

If you want then to describe an action/motion, you can do it in two ways:

- Picking a point, here there is a screw: an axis and a pitch → **geometric description**: (ℓ, h)
- Using two vectors and a point O: with shifting law you can find action/motion for every point O' → **algebraic description**: $[\vec{f}, \vec{m_O}]$ or $[\vec{\omega}, \vec{v_O}]$

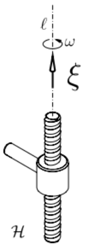
We can shift from one representation to another:

- if we know $\vec{f}, \vec{m_O}$ / $\vec{\omega}, \vec{v_O}$ and O with formulas we can compute h in order to get the pitch and $\vec{r}_\perp$ in order to find the axis ($\ell \perp \vec{r}_\perp$)
- if we know the screw pitch h and position of the axis we can find $\vec{f}, \vec{m_O}$ / $\vec{\omega}, \vec{v_O}$ at O

<u>NOTE:</u> a screw axis (a line) is expressed with a point and a direction vector $\{P + \lambda\vec{u} | \lambda \in \mathbb{R}\}$

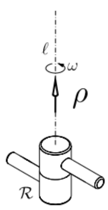
2.4 Summary on ξ and ζ

For pure rotations and pure translations, we indicate the twists with ρ and τ:



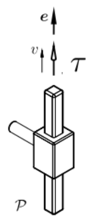
$$\xi = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix} = \omega \begin{bmatrix} \boldsymbol{e} \\ h\boldsymbol{e} + \boldsymbol{r} \times \boldsymbol{e} \end{bmatrix}$$

$\omega \neq 0, h \neq 0$



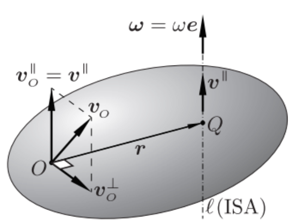
$$\rho = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix} = \omega \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{r} \times \boldsymbol{e} \end{bmatrix}$$

$\omega \neq 0, h = 0$

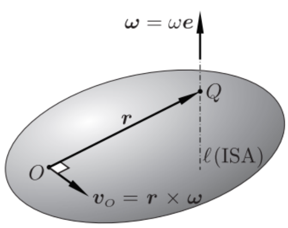


$$\tau = \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{v}_O \end{bmatrix} = v \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{e} \end{bmatrix}$$

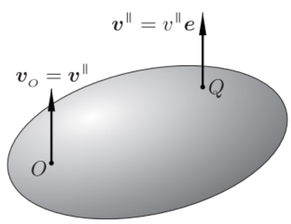
$\omega = 0, h = \infty$



$$\boldsymbol{v}_O = \underbrace{\boldsymbol{v}^{\parallel}}_{\parallel \boldsymbol{e}} + \underbrace{\boldsymbol{\omega} \times \boldsymbol{r}}_{\perp \boldsymbol{e}}$$
$$= \omega(h\boldsymbol{e} + \boldsymbol{e} \times \boldsymbol{r})$$

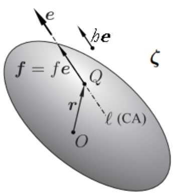


$\boldsymbol{v}^{\parallel} = \boldsymbol{0}$



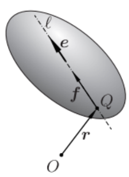
$\boldsymbol{\omega} = \boldsymbol{0}$

For pure forces and pure couples, we indicate the wrenches with φ and μ:



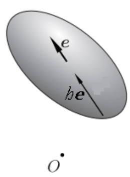
$$\zeta = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{m}_O \end{bmatrix} = f \begin{bmatrix} \boldsymbol{e} \\ h\boldsymbol{e} + \boldsymbol{r} \times \boldsymbol{e} \end{bmatrix}$$

$f \neq 0, h \neq 0$



$$\varphi = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{m}_O \end{bmatrix} = f \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{r} \times \boldsymbol{e} \end{bmatrix}$$

$f \neq 0, h = 0$



$$\mu = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{m}_O \end{bmatrix} = f \begin{bmatrix} 0 \\ h\boldsymbol{e} \end{bmatrix}$$

$f = 0, h = \infty$

Exercises

1. The axis ℓ of a unit twist passes from $(1, 0, 0)$ to $(0, 0, 1)$. Find the velocity at O if $h = 1$.
2. The axis ℓ of a unit-amplitude twist of pitch h is horizontal at angle θ to O_x through $(0, 0, 1)$. Find the vector components of the twist.
3. Find the axis and the pitch of the wrench $(-\vec{i} + \vec{j}, -2\vec{i} + 2\vec{k})$. Make a clear drawing showing the axes location in 3D.

DO THEM!

2.5 Linear combinations & span

The **span** of $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n \in V$ is the set of their linear combinations:

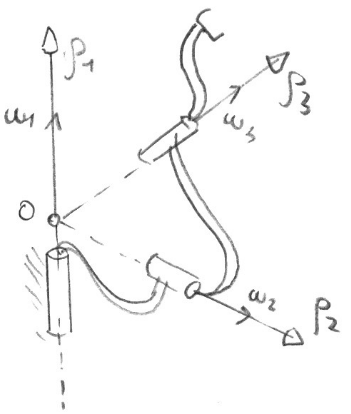
$$Span(\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n) = \{\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + ... + \lambda_n \boldsymbol{v}_n | \lambda_i \in \mathbb{R}\}$$

An example of span of two vectors is the plane containing them in the 3D space.

But what si the interpretation of the span regarding instantaneous kinematics and statics?

- $Span\{twists\}$: all end-effector **motions** of a serial chain ???
- $Span\{wrenches\}$: all end-effector **constraints** of a parallel chain ???

Exercise Consider a **spherical** RRR chain (concurrent axes). Find all possible instantaneous motions of the end-effector and the second link.



We have only pure rotations $h = 0$ and the axis passes through O ($|\vec{r}| = 0$) so:

$$\xi_i \rightarrow \text{ pure rotation } \rightarrow \rho_i = (\vec{\omega_i}, 0 \cdot \vec{\omega_i} + \vec{r} \times \vec{\omega_i}) = (\omega_i \vec{u}_i, \vec{0})$$

where $\vec{u}_i$ is the unit vector along the rotation axis and the $\vec{0}$ is because in this case (**spherical** configuration) the **velocity in** O in **null** because O is along the axis of each rotation.

So:

$$\xi = \rho_1 + \rho_2 + \rho_3 = (\omega_1 \vec{u_1} + \omega_2 \vec{u_2} + \omega_3 \vec{u_3},\ \vec{0}) = (\vec{\omega}, \vec{0})$$

<u>Case 1:</u> $\vec{\omega} = \omega_1 \vec{u_1} + \omega_2 \vec{u_2} + \omega_3 \vec{u_3} = 0$ if $(\omega_1, \omega_2, \omega_3) = (0, 0, 0)$. In this case we have:

$$h = \frac{\vec{\omega} \cdot \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} = \infty \qquad \xi = (\vec{0}, \vec{0})$$

So $h = \infty$ but, since $\vec{v_O} = \vec{0}$, the end-effector **cannot translate**:

$$\xi = \rho_1 + \rho_2 + \rho_3 = (\vec{0}, \vec{0})$$

<u>Case 2:</u> Now consider the case $\vec{\omega} \neq 0$. We know that $\vec{\omega} \cdot \vec{v_O} = 0$. From this we can obtain:

$$h = \frac{\vec{\omega} \cdot \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} = 0$$

So $h(\xi) = 0 \rightarrow$ the end effector can **only rotate**:

$$\xi = \rho_1 + \rho_2 + \rho_3 = (\omega_1 \vec{u_1} + \omega_2 \vec{u_2} + \omega_3 \vec{u_3},\ \vec{0})$$

We also know that $\vec{\omega} \times \vec{v_O} = 0$. From this we can obtain:

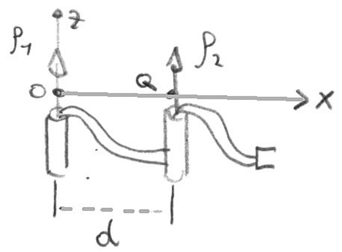
$$\vec{r}_\perp = \frac{\vec{\omega} \times \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} = \vec{0}$$

We obtained also $\vec{r}_\perp = \vec{0} \rightarrow$ resultant screw's axis passes through O. The direction of the axis defined by $\vec{\omega} = \vec{\omega_1} + \vec{\omega_2} + \vec{\omega_3}$. So, if $\vec{u_1}, \vec{u_2}, \vec{u_3}$ aren't coplanar (that is this case), axis may have **any direction** but it **passes through** O.

We can conclude that all the allowed motions are all the rotations around any axis passing through O, that is:

$$\boxed{J = Span(\rho_1, \rho_2, \rho_3) = \text{ all rot. around axis passing through } O}$$

Exercise Consider a planar RR chain. Find all instantaneous motions of the end-effector.



Start considering the the pure rotation twist of the first R joint ρ_1:

$$\rho_1 = (\omega_1 \vec{u}_1, \vec{v_O}) = (\omega_1 \hat{k}, \vec{0})$$

where $\vec{u}_1 = \hat{k}$ is the unit vector along the axis z (rotation axis) and the $\vec{0}$ is because chosen O along z so the velocity $\vec{v_O} = (\vec{0} + \vec{r} \times \vec{\omega})$ is null.

Considering both joints, we will have $\vec{u}_i = \hat{k}$ since it is a **planar** configuration and, considering a point Q along the axis of rotation ρ_2, we will be able to express:

$$\vec{v_Q} = \vec{\omega} \times \vec{QO} = \vec{OQ} \times \vec{\omega} \sim \vec{OQ} \times \hat{k}$$

$$\rho_2 = (\omega_2 \vec{u}_2, \vec{v_Q}) = (\omega_2 \hat{k}, \omega_2 \vec{OQ} \times \hat{k})$$

$$\xi = \rho_1 + \rho_2 = \Big[(\omega_1 + \omega_2)\hat{k},\ \vec{r} \times (\omega_2\ \vec{OQ} \times \hat{k})\Big]$$

<u>Case 1:</u> $\vec{\omega} = (\omega_1 + \omega_2)\hat{k} = 0$ if $\omega_1 = -\omega_2$. In this case, we have:

$$h = \frac{\vec{\omega} \cdot \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} = \infty \qquad \xi = \Big[\vec{0},\ \vec{r} \times (\omega_2\ \vec{OQ} \times \hat{k})\Big]$$

So $h = \infty$ and then the end-effector **can translate** in direction $\vec{OQ} \times \hat{k} = d(\hat{\imath} \times \hat{k}) \sim -\hat{\jmath}$ (where d is the distance between O and Q), so **along the *y* axis**, as we can analytically see below:

$$\xi = \rho_1 + \rho_2 = \Big[\vec{0},\ \vec{r} \times (\omega_2\ \vec{OQ} \times \hat{k})\Big] = \Big(\ \vec{0},\ \omega_1\ d\ \hat{\jmath}\ \Big)$$

<u>Case 2:</u> If we have instead $\vec{\omega} \neq 0$, then:

- It **can rotate** (along $\hat{k}$ according to its twist), so $h(\xi) = 0$. In fact, analytically we notice that in this case :

$$\vec{\omega} \cdot \vec{v} \sim \hat{\imath} \cdot \hat{\jmath} = 0$$

- It rotates with vertical axes ($/\!\!/O_z$). In fact if we compute the perpendicular vector, we have:

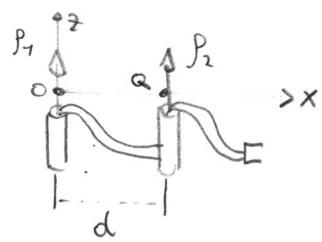
$$\vec{r_\perp} = \frac{\vec{\omega} \times \vec{v_O}}{\vec{\omega} \cdot \vec{\omega}} = \frac{\cancel{(\omega_1 + \omega_2)}\omega_2\, d}{\cancel{(\omega_1 + \omega_2)}(\omega_1 + \omega_2)}\hat{i} = \frac{d\, \omega_2}{(\omega_1 + \omega_2)}\hat{i}$$

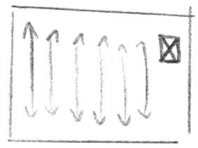
  So we know that $\vec{r_\perp} \sim \hat{i}$, so the axis **intersect the x axis**. Moreover, since d is constant and $\frac{\omega_2}{(\omega_1+\omega_2)}$ can be any number depending on the values on ω_1 and ω_2, the rotation axis is along $\hat{k}$ that can intersect the x axis **in every point**.

So, in the end:

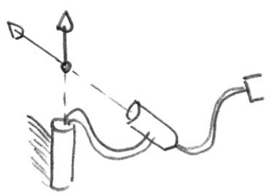
$$\boxed{J = Span(\rho_1, \rho_2) = \begin{cases} \text{all rotations around axis } /\!\!/ \text{ to } z \text{ passing through any point of } x \\ \text{and any translation along axis } /\!\!/ \text{ to } y \end{cases}}$$

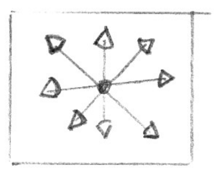
So, if we compare the RR planar chain with the respective span:





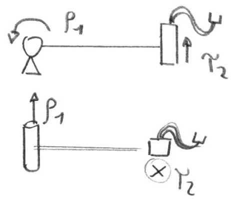
with the previous RR "intersecting-axis" chain:





we can notice the same differences and analogies seen in the introduction, regarding the possible combinations generated by the configuration.

But what if we put a P-joint instead of the second R-joint?



What are the allowed motions of the end effector now? What is the *Span*?

$$J = Span(\rho_1, \tau_2) =?$$

- Before we had $\omega_1\rho_1 + \omega_2\rho 2$
- Now we have $\omega_1\rho_1 + v_2\tau 2$

→ but wait! This **translation** along an axis $/\!\!/$ to y **is already in** the previously found *Span*!

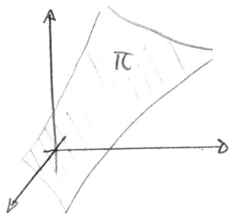
$$\xi = \rho_1 + \rho_2 = \left[(\omega_1 + \omega_2)\hat{k},\ \omega_2\ (\vec{OQ} \times \hat{k})\right]$$

2.6 Linear subspaces

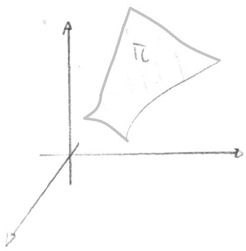
W is a **linear subspace** of the vector space V if:

- W is a subset of V
- W is a vector space with VA and SM of V

An example of linear subspace is a plane passing through the origin:

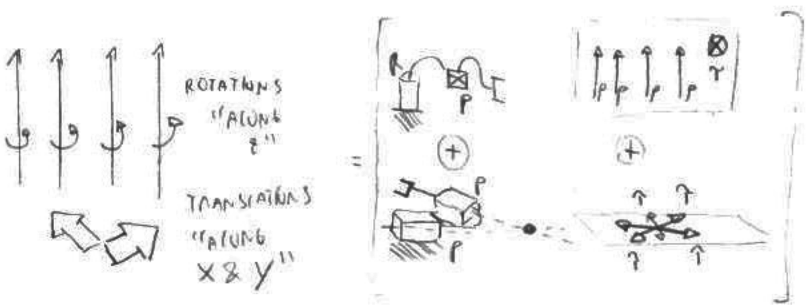


A plane not through the origin instead is not a linear subspace.



A subspace **inherits the zero** $\rightarrow$ be sure you have it (it's an easy check).

An example of linear subspaces are also the **planar motions** $se(2) \subset se(3)$:



– **Theorem**:

The **intersection** of two subspaces is a **subspace**.
The non-trivial **union** of two subspaces is **not a subspace**.
The **difference** of two subspaces is **never** a subspace (you always subtract $\mathbf{0}$).

– **Theorem**:

$Span(\vec{v_1}; \vec{v_2}, ..., \vec{v_n})$ is the smallest subspace with $\{\vec{v_1}; \vec{v_2}, ..., \vec{v_n}\}$

An example for the first theorem are the **impossible motions** (for example the ones of a planar-chain end-effector).

Let's consider a serial chain that has only R and P joints. At some configuration, instantaneously, the end-effector:

- can rotate about O_z
- cannot rotate about O_x
  cannot rotate about O_y
  cannot translate in x
  cannot translate in y
  cannot translate in z

What is the (instantaneous) DoF of the end-effector? Can you draw the serial chain? Is your solution of the previous questions the only one possible? If no, what alternatives are there?

If the twist $\rho = (0, \hat{i})$ is not allowed, the rotation along axis x is not allowed, but only on that axis! It means that the velocity vectors are not along the forbidden axis, but they can have components along them $\rightarrow$ **the five cannot** are just noise! They don't help us, **they confuse us**.

If you pick 6 random motions they will surely form a base. In fact, it is impossible that, if randomly chosen, they will be linear dependent, because of the dense nature of space: if you chose three random points, in they are random, they will never be on a single line.

So, for a serial chain that has only R and P joints at some configuration, instantaneously, the end-effector can have the set of twists Y and cannot have the set of twists N, the (instantaneous) DoF of the end-effector are the one defined by Y, because it is the only set of twist that carries some information.

2.7 Linear independence

We have to introduce a different definition w.r.t the common "operative" definition of linear independence, that is, "a vector is linear independent w.r.t. a set of other vectors if it cannot be expressed as linear combination of the others in the set". The more "technical" definition is the following one:

The set $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ is **linearly dependent** if:

$$\exists(\lambda_1, \lambda_2, ..., \lambda_n) \neq (0, ..., 0) \text{ s.t. } \lambda_1\vec{v_1} + \lambda_2\vec{v_2} + ... + \lambda_n\vec{v_n} = 0$$

Else it's linear independent.

When two **twists** are linearly dependent? When they have the **same axis** but not only! They have to have the **same pitch**! (e.g. Two rotations are linearly independent if they are in the same axis but with different pitch)

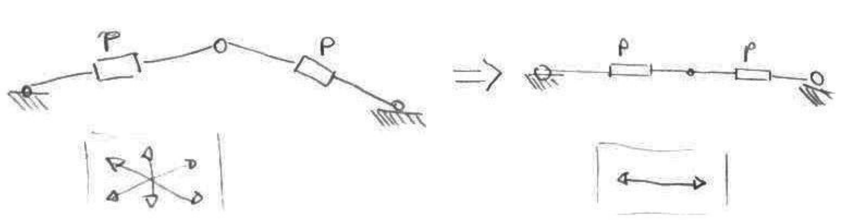
Three facts about linear dependence:

- subset of a linearly **indep**endent set is linearly **indep**endent
- superset of a linearly **dep**endent set is linearly **dep**endent
- If $\vec{0} \in \{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ then $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_n}\}$ are linearly **dep**endent

W.r.t the third point, consider this case:

$$\left[\underset{\in se(3)}{\xi} = \lambda_1 \underset{\in se(3)}{\xi_1} + \lambda_2\xi_2 + ... + \lambda_n\xi_n = \vec{0}\right] \Rightarrow \text{ linearly } \textbf{dep}\text{endent}$$

so, if they are linearly dependent, there is a non-zero motion that maintain zero the total motion, because **having a $\vec{0}$** as resultant $\xi = \vec{0}$ means that you can hold the end effector but it can move anyway $\rightarrow$ **singularity**. Let's analyse this singularity case with 2 activated P-joints:



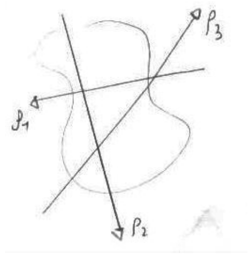
Here, $\xi = \tau_1 + \tau_2 = \vec{0}$ means that, with certain $v_{1,2}$ values $\{v_1, v_2 = -v_1\}$ the span of the parallel manipulator is reduced as seen (only along the line $\rightarrow v_1 + v_2 \in v_{1,2}$ axis): it is the singular configuration.

In the non-singular configuration, $\rightarrow v_1 + v_2 \notin v_1$ axis and $\rightarrow v_1 + v_2 \notin v_2$ axis $\rightarrow \xi = \tau_1 + \tau_2 \neq \vec{0}$.

Then, formally and w.r.t. the third point seen above, we can say, for the singular configuration:

$$\xi_{singular} = \tau_1 + \tau_2 = (0, v_1\vec{x} + v_2\vec{x}) = 0 \quad \text{for } v_1 = v, \;\; v_2 = -v \quad \Rightarrow \exists \boldsymbol{v} = \{v, -v\} \text{ s.t. } \xi = \vec{0}$$

Consider three random **forces on a rigid body**:



They are linearly independent because, given

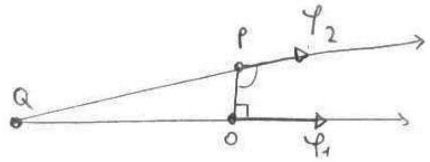
$$\rho_1 = (\vec{u_1}, 0) \qquad \rho_2 = (\vec{u_2}, 0) \qquad \rho_3 = (\vec{u_3}, \vec{r} \times \vec{u_3})$$

we notice that ρ_3 cannot be expressed as a linear combination of ρ_1 and ρ_2.

In general we will have:

Rotations		
Planar	parallel	$\rightarrow \rho_1, \rho_2$ linearly indep.
	through the same point	$\rightarrow \rho_1, \rho_2$ linearly indep.
3D space	through the same point	$\rightarrow \rho_1, \rho_2, \rho_3$ linearly indep.
Translations		
Planar	parallel	$\rightarrow \tau_1, \tau_2$ linearly indep.
	through the same point	$\rightarrow \tau_1, \tau_2$ linearly indep.
3D space	through the same point	$\rightarrow \tau_1, \tau_2, \tau_3$ linearly indep.

If we consider the pure forces:



$$\varphi_1 = (\vec{f_1}, 0), \qquad \varphi_2 = (\vec{f_2}, \vec{OP} \times \vec{f_2})$$

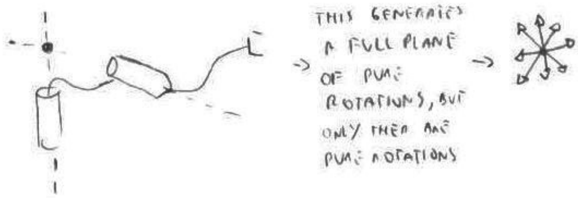
if we try to express the third force as a linear combination of φ_1 and φ_2:

$$\varphi_3 = a_1\varphi_1 + a_2\varphi_2 = (a_1\vec{f_1} + a_2\vec{f_2}, a_2\vec{OP} \times \vec{f_2})$$

$$h = \frac{(a_1\vec{f_1} + a_2\vec{f_2}) \cdot (a_2\vec{OP} \times \vec{f_2})}{\dots} = \frac{a_1 a_2 \vec{f_1}\vec{OP} \times \vec{f_2}}{\dots}$$

In this case, we will have $h = 0$ if and only if $a_1 = a_2 = 0$, that is because **the linear comb. of two pure forces in the plane will not be a pure force!** $\rightarrow$ so, if φ_3 is a pure force and does not pass from Q, it is linear independent!

We have the same for twists:



$\rho \in T \leftarrow$ *pure rotation* $\qquad \rho \notin T \rightarrow$ *not a pure rotation*

2.8 Basis, dimension & singularities

Definitions:

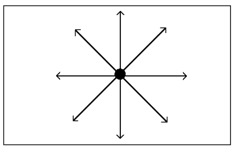
$$\begin{array}{lll} \dim(V) < \infty & \textbf{\textit{if}} & V = Span = (\vec{v_1}, ..., \vec{v_n}) \\ \dim(V) = n & \textbf{\textit{if}} & \exists\{\vec{v_1}, ..., \vec{v_n}\} \text{ lin. indep. (a basis) s.t. } V = Span = (\vec{v_1}, ..., \vec{v_n}) \end{array}$$

Proposition:

$$\begin{array}{lll} \dim(V) = n & \Leftrightarrow & \exists n\text{-basis} \quad \Leftrightarrow \quad \forall \text{basis is a } n\text{- basis} \\ & \Leftrightarrow & \forall \text{ if } \{\vec{v_1}, ..., \vec{v_n}\} \text{ lin. indep. then } \{\vec{u}, \vec{v_1}, ..., \vec{v_n}\} \text{ is lin. dep. } \forall \vec{u} \end{array}$$

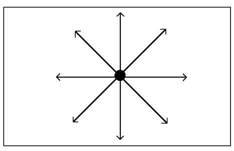
Exercise: Find a basis and establish the dimension of:

- The vector space of **rotations** with axes through a point O:



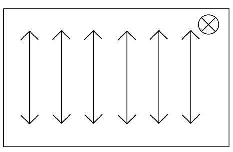
  This is a vector space of dim $= 2$ because it can be generated by two rotations passing through the point (as already seen). So, a base can be $\{\rho_1, \rho_2\}$ with ρ_1, ρ_2 passing through the point O.

- The vector space of **translations** with axes through a point O:



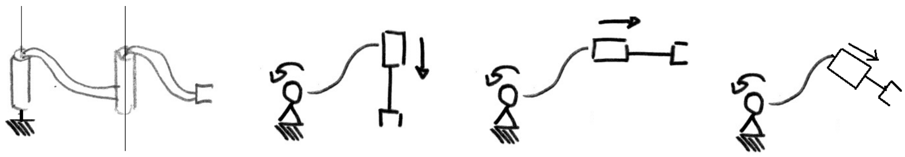
  This is a vector space of dim $= 2$ because it can be generated by two translations passing through the point. So, a base can be $\{\tau_1, \tau_2\}$ with τ_1, τ_2 passing through the point O.

- The "$/\!/$ **rotations** plus $\perp$ **translation**" vector space:

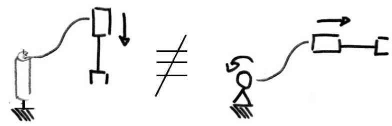


  This is a vector space of dim $= 2$ because it can be generated by two parallel rotations (as already seen). So, a base can be $\{\rho_1, \rho_2\}$ with ρ_1, ρ_2 parallels, but we have already seen that it can also be generated by a rotation and a $\perp$ translations, so other equivalent base will be $\{\rho_1, \tau_2\}$ and $\{\tau_1, \rho_2\}$ with $\rho \perp \tau$.

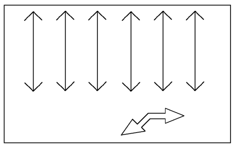
In fact, all these are equivalent manipulators: [1]



Be careful! When we go in 3D is not the same:

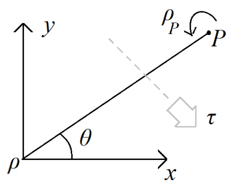


- The **planar motions** $se(2)$:



This is a vector space of $\dim[se(2)] = 3$ because it can be generated by two translations passing through a point O and a rotation $\perp$ to the plane defined by the two translations. So, a base can be $\{\tau_1, \tau_2, \rho_3\}$ with τ_1, τ_2 passing through the point O and $\rho_3 \perp \tau_{1,2}$.

But how can we really generate any rotation around P in the plane xy defined by the two translations, given only a rotation along the z axis?



As seen in figure, we need a translation $\tau \perp$ to the position vector of P (from ρ_z to ρ_P) and the rotation ρ in order to generate the "$/\!\!/$ rotations plus $\perp$ translation" vector space to which ρ_P belongs. Since we have τ_x and τ_y, we can create any τ on the plane, so that we will be able to generate any rotation $\perp$ to this plane:

$$\rho_P \in Span(\rho_z, \tau), \qquad \tau = \sin(\theta)\tau_x - \cos(\theta)\tau_y$$

Another way to generate it is the following one:

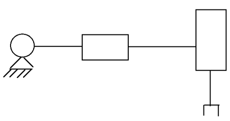
$$\rho_P = (\vec{k}, ?), \qquad \vec{v_O} = \vec{OP} \times \vec{\omega} = \vec{OP} \times \vec{k} = y_P\vec{i} - x_P\vec{j}$$

$$\rho_P = (\vec{k}, y_P\vec{i} - x_P\vec{j}), \qquad \tau_x = (0, \vec{i}), \;\; \tau_y = (0, \vec{j}), \;\; \rho_z = (\vec{k}, \vec{0})$$

From this last expression we can clearly see how may times we have to take them in the linear combination:

$$\rho_P = \rho_z + y_P\tau_x - x_p\tau_y$$

A possible physical realization will be:



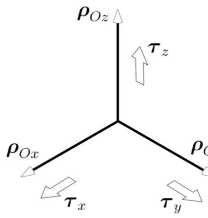
[1] In Figure only RR and 3 RP, but also PR are equivalent, they just need $\perp$ joint axis.

Following what we said in this last point, we can notice that a **planar PPP** chain will result in a **singularity** because we cannot have a planar motion base with 3 planar translations:

$$\{\tau_x, \tau_y, \tau_{\in Span(\tau_x,\tau_y)}\} \rightarrow \dim = 2$$

But before starting the singularity analysis introducing the Jacobian, let's see some **standard bases**:

- The **Plücker bases** for a frame $Oxyz$ is the standard bases for a 3D space.
  - For the **twists**, and has the 3 unit rotations about the axes and the 3 unit translations directed as the axes:



$$\{\rho_{Ox},\ \ \rho_{Oy},\ \ \rho_{Oz},\ \ \tau_x,\ \ \tau_y,\ \ \tau_z\}$$

$\rightarrow$ with this class of coordinates the twist will be expressed like this:

$$\xi = (\vec{\omega}, \vec{v_O}) = (\omega_x\vec{i} + \omega_x\vec{j} + \omega_x\vec{k} + v_{Ox}\vec{i} + v_{Oy}\vec{j} + v_{Oz}\vec{k}) =$$

$$= \omega_x\rho_{Ox} + \omega_x\rho_{Oy} + \omega_x\rho_{Oz} + v_{Ox}\tau_x + v_{Oy}\tau_y + v_{Oz}\tau_z$$

  - Same for the **wrenches**, is composed by the 3 unit forces along the axes and by the 3 unit couples directed as the axes:

$$\{\varphi_{Ox},\ \ \varphi_{Oy},\ \ \varphi_{Oz},\ \ \mu_x,\ \ \mu_y,\ \ \mu_z\}$$

$\rightarrow$ with this class of coordinates the twist will be expressed like this:

$$\zeta = (\vec{f}, \vec{v_O}) = (f_x\vec{i} + f_x\vec{j} + f_x\vec{k} + m_{Ox}\vec{i} + m_{Oy}\vec{j} + m_{Oz}\vec{k}) =$$

$$= f_x\varphi_{Ox} + f_x\varphi_{Oy} + f_x\varphi_{Oz} + m_{Ox}\mu_x + m_{Oy}\mu_y + m_{Oz}\mu_z$$

[**beware** of unit problems! Are, for example, fx, mx dimensioned? Later we will talk about it]

- Another bases that can be used is the **Ball bases** for a frame $Oxyz$. In this case, instead of talking about 0-pitch screws and ∞-pitch screws along the axis (Plücker) we can take ± 1-pitch screws along the three axis:

$$\{\xi^+_{Ox},\ \ \xi^+_{Oy},\ \ \xi^+_{Oz},\ \ \xi^-_{Ox},\ \ \xi^-_{Oy},\ \ \xi^-_{Oz}\}$$

Exercise: The directed axis ℓ is along the segment from point $(1, 0, 0)$ to point $(0, 1, 1)$. Find the Plücker coordinates of these twists:

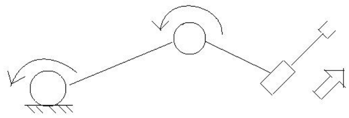
1. A unit pure clockwise rotation about ℓ.
2. A unit pure counter-clockwise rotation about ℓ.
3. A unit translation directed as ℓ.
4. A twist on ℓ with pitch $-1/2m$ and amplitude $2rad/s$.

DO THEM!

So, we noticed that a **planar PPP** chain will result in a **singularity** because of the impossibility to have a planar motion base with 3 planar translations:

$$\{\tau_x, \tau_y, \tau_{\in Span(\tau_x, \tau_y)}\} \rightarrow \dim = 2$$

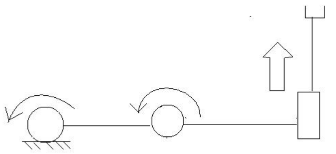
But can we have three joints in a planar chain in order to obtain a space which dimension is 3 (and so it has three screws as basis)? Let's consider the following example:



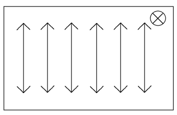
$$\{\rho_1, \rho_2, \tau_3\}$$

Is this ok? Does it form a sound basis (a.k.a. are they linearly independent)? In general "yes", but consider the following three cases:

- Case 1:

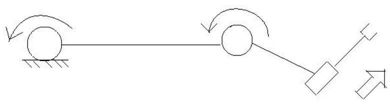


The translation τ_3 is $\perp$ **to the axis** passing through the points of ρ_1 and ρ_2 $\rightarrow$ it's a **singular configuration**: there is a "$/\!\!/$ **rotations** plus $\perp$ **translation**" vector space generated by ρ_1 and ρ_2, but **the same** is also generated by ρ_1 and τ_3 and ρ_2 and τ_3:

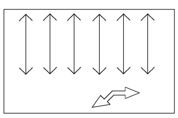


$$\dim = 2 \quad , \qquad \text{possible base:}\{\rho_1, \rho_2\} \text{ or}\{\rho_1, \tau_3\} \text{ or}\{\rho_2, \tau_3\}$$

- Case 2:

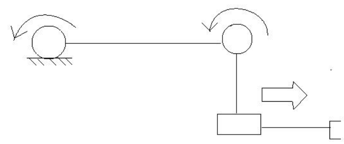


The translation τ_3 is **not** $\perp$ **to the axis** passing through the points of ρ_1 and ρ_2 $\rightarrow$ it's not a singular configuration: there is a "$/\!\!/$ **rotations** plus $\perp$ **translation**" vector space generated by ρ_1 and ρ_2 and **another, different one** generated by ρ_2 and τ_3, and together they form $se(2)$ (see later about sum of vector spaces):



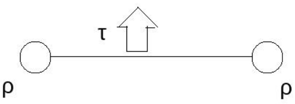
$$\dim = 3 \quad , \qquad \text{possible base:}\{\rho_1, \rho_2, \tau_3\}$$

- Case 3 (if you notice, is the same as case 2):

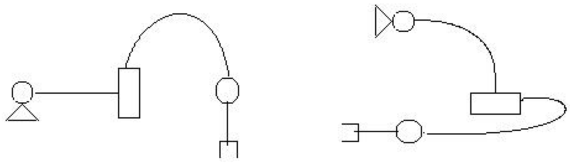


So, in a singular configuration, the chain **loses one instantaneous DoF** or, better, the motion space **loses one dimension**.

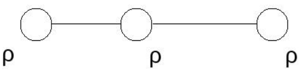
With a RRP/PRR/RPP planar chain we will have a singular configuration only when:



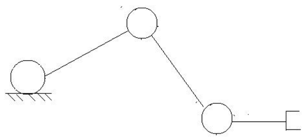
like in these situations:



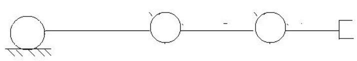
Can we have a basis formed by three rotations $\{\rho_1, \rho_3, \rho_3\}$? Yes, the only "bad" configuration is:



So, the following serial planar manipulator's motion space is $se(2)$ (dim = 3):

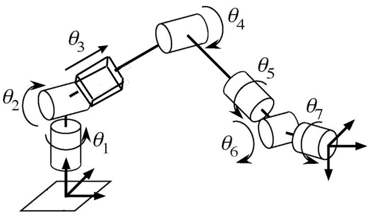


but its singular configuration's motion space is only the "// **rotations** plus ⊥ **translation**" vector space (dim = 2):



2.9 Jacobian matrix

So, now, let's consider the following RRPRRRR serial chain:



$$\{\rho_1,\ \rho_2,\ \tau_3,\ \rho_4,\ \rho_5,\ \rho_6,\ \rho_7\ \}$$

$$\xi = \lambda_1\rho_1 + \lambda_2\rho_2 + \lambda_3\tau_3 + \lambda_4\rho_4 + \lambda_5\rho_5 + \lambda_6\rho_6 + \lambda_7\rho_7 \quad (\lambda_i \textbf{ dimensionless})$$

$$\xi = \lambda_1 \begin{bmatrix} \vec{e_1} \\ \vec{r_1} \times \vec{e_1} \end{bmatrix} + \lambda_2 \begin{bmatrix} \vec{e_2} \\ \vec{r_2} \times \vec{e_2} \end{bmatrix} + \lambda_3 \begin{bmatrix} \vec{0} \\ \vec{e_3} \end{bmatrix} + \lambda_4 \begin{bmatrix} \vec{e_4} \\ \vec{r_4} \times \vec{e_4} \end{bmatrix} + \lambda_5 \begin{bmatrix} \vec{e_5} \\ \vec{r_5} \times \vec{e_5} \end{bmatrix} + \lambda_6 \begin{bmatrix} \vec{e_6} \\ \vec{r_6} \times \vec{e_6} \end{bmatrix} + \lambda_7 \begin{bmatrix} \vec{e_7} \\ \vec{r_7} \times \vec{e_7} \end{bmatrix}$$

$$\xi = J\dot{\boldsymbol{\theta}} \qquad \dot{\boldsymbol{\theta}} = (\omega_1, \omega_2, \tau_3, \omega_4, \omega_5, \omega_6, \omega_7) \qquad \Big(\xi_{(6\times1)},\ J_{(6\times7)},\ \dot{\boldsymbol{\theta}}_{(7\times1)}\Big)$$

!!!: be careful! $\dot{\boldsymbol{\theta}}$'s elements are the **joint speeds**, they are not dimensionless, they are expressed in units of measure, wile the **columns** of the Jacobian J are **not twists**, they are not vectors in the same vector space. This because, since λ_i are dimensionless (note: rad are dimensionless as well):

$$\lambda_i \begin{bmatrix} \vec{0} \\ \vec{e_i} \end{bmatrix} \dot{\theta}_i + \lambda_i \begin{bmatrix} \vec{e_j} \\ \vec{r_j} \times \vec{e_j} \end{bmatrix} \dot{\theta}_j \quad \xrightarrow{if\ twists} \quad [\,\cdot\,] \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix} [rad/s] + [\,\cdot\,] \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix} [m/s] \rightarrow \textbf{not dimensionally sound !}$$

That is because, in linear algebra we don't care about the units of measure of $\dot{\boldsymbol{\theta}}$, and we can consider them as simple numbers, and the vectors $\vec{e}$ will be the velocities:

$$\text{Linear algebra approach:} \quad [\,\cdot\,] \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix} [\cdot] + [\,\cdot\,] \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix} [\cdot] = \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix}$$

But since we are interested in the **physical meaning** of $\dot{\boldsymbol{\theta}}$ (the joint speeds vector), then we cannot consider the J columns as twists. Then the vectors $\vec{e}$ will be the simple directions, and the units of measure will have to be adapted with a **conversion matrix**, that has the values of the identity matrix but appropriated units of measure:

$$\text{Physical appr. } (\rho_i): \quad \lambda_i \begin{bmatrix} \vec{e_i} \\ \vec{r_i} \times \vec{e_i} \end{bmatrix} \dot{\theta}_i \;=[\,\cdot\,] \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ m/rad \\ m/rad \\ m/rad \end{bmatrix} [rad/s] \;=\; \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix} \leftrightarrow \begin{bmatrix} \vec{e_i} \\ \vec{r_i} \times \vec{e_i} \end{bmatrix} = \rho_i \begin{bmatrix} 1\ s/rad \\ 1\ s/rad \\ 1\ s/rad \\ 1\ s/rad \\ 1\ s/rad \\ 1\ s/rad \end{bmatrix}$$

$$\text{Physical appr. } (\tau_i): \quad \lambda_i \begin{bmatrix} \vec{0} \\ \vec{e_i} \end{bmatrix} \dot{\theta}_i \;=[\,\cdot\,] \begin{bmatrix} rad/m \\ rad/m \\ rad/m \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} [m/s] \;=\; \begin{bmatrix} rad/s \\ rad/s \\ rad/s \\ m/s \\ m/s \\ m/s \end{bmatrix} \longleftrightarrow \begin{bmatrix} \vec{0} \\ \vec{e_i} \end{bmatrix} = \tau_i \begin{bmatrix} 1\ s/m \\ 1\ s/m \\ 1\ s/m \\ 1\ s/m \\ 1\ s/m \\ 1\ s/m \end{bmatrix}$$

Our conversion matrix for our RRPRRRR will be then:

$$\begin{bmatrix} 1\ s/rad & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1\ s/rad & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1\ s/m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1\ s/rad & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1\ s/rad & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1\ s/rad & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1\ s/rad \end{bmatrix}$$

And the Jacobian units of measure will be:

$$[J] = \begin{bmatrix} \cdot & \cdot & rad/m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & rad/m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & rad/m & \cdot & \cdot & \cdot & \cdot \\ m/rad & m/rad & \cdot & m/rad & m/rad & m/rad & m/rad \\ m/rad & m/rad & \cdot & m/rad & m/rad & m/rad & m/rad \\ m/rad & m/rad & \cdot & m/rad & m/rad & m/rad & m/rad \end{bmatrix}$$

Then, treat the Jacobian matrix with care! Also for an operation like $J^T J$, we are not allowed to do it if we don't interpose a conversion matrix, because (we got rid of rad since are dimensionless):

$$[J^T T J] = \begin{bmatrix} \cdot & \cdot & \cdot & m & m & m \\ \cdot & \cdot & \cdot & m & m & m \\ 1/m & 1/m & 1/m & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & m & m & m \\ \cdot & \cdot & \cdot & m & m & m \\ \cdot & \cdot & \cdot & m & m & m \\ \cdot & \cdot & \cdot & m & m & m \end{bmatrix} T \begin{bmatrix} \cdot & \cdot & 1/m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/m & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/m & \cdot & \cdot & \cdot & \cdot \\ m & m & \cdot & m & m & m & m \\ m & m & \cdot & m & m & m & m \\ m & m & \cdot & m & m & m & m \end{bmatrix}$$

So, be careful: when you will search the **singular values** of JJ^T, they will have a meaning only w.r.t their zero or non-zero value (in order to identify a singularity), but their value is inconsistent, because of the units of measure differences between elements ($\rightarrow$ it is not true that "the more it is high, the more we're far from the singularity", or better, it is true, but we don't know the scale, so it could be instead ridiculously near!).

2.10 Sums and direct sums of subspaces

Definition of $+$: let $U, W \subset V$ (subspaces). Then:

$$U + W = \{\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \mid \boldsymbol{u} \in U,\ \boldsymbol{w} \in W\}$$

From this we can deduce that:

$$U + W = Span(U \cup W)$$

$$\boxed{\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)} \leftarrow \text{(important!!)}$$

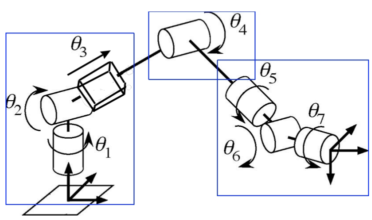
Definition of $\oplus$: let U, W two vector spaces. Then:

$$V = U \oplus W \iff [(V = U + W) \wedge (U \cap W) = \emptyset]$$

Some examples :

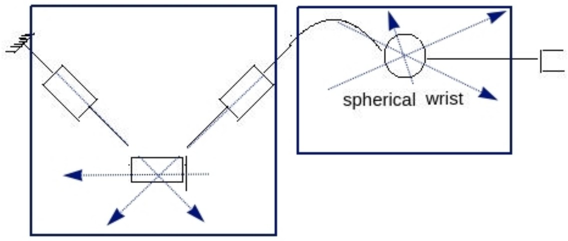
- Two planes in 3D: $\oplus$ if they are parallel
- A line and a plane in 3D: $\oplus$ if there is no intersection point (they are parallel)
- Two motion spaces: $\oplus$ if there is no motion in common (see below)

Sometimes is useful to split the motion space analysis in parts, and then sum the subspaces in order to have the full motion space of the manipulator:



$$V = Span\{\rho_1, \rho_2, \tau_3\} + Span\{\rho_4\} + Span\{\rho_5, \rho_6, \rho_7\}$$

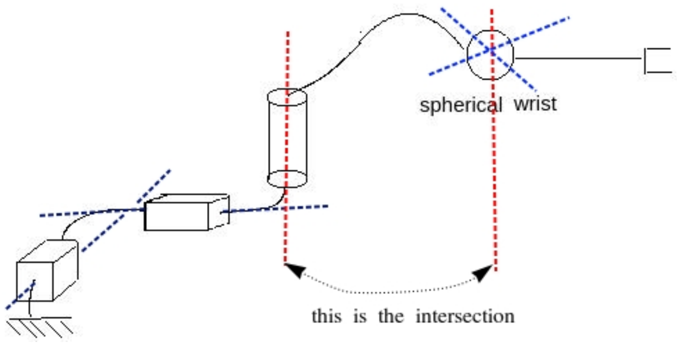
Considering an easier or example, we can have spherical translations subset and a spherical rotations subset like the following one:



In this case, there is **no intersection, no simplification**, and the singularities will be the ones of the two parts:

$$Total\ Span = Span\{\tau_1, \tau_2, \tau_3\} \oplus Span\{\rho_4, \rho_5, \rho_6\}$$

We can also have planar motions ($se(2)$) combined with spherical rotations like in the following case:



In this case, the ρ_z spanned by the spherical joint is the same motion in $se(3)$ spanned by the PPR subsystem, so we have an **intersection**:

$$Total\ Span = Span\{\tau_1, \tau_2, \tau_3\} + Span\{\rho_4, \rho_5, \rho_6\} \qquad Total\ \dim = 3 + 3 - 1$$

This is very useful in constraint analysis of serial chains → in it you'll have planar and spherical subchains, and you will just have to keep in mind their basis.

Some subspaces are easier, some aren't: 3 randomly chosen rotations makes a subspace but it is very difficult to understand what they generate → in 6 dimensions is a plane, but squeezed in our 3-dimensional world it is not as intuitive as the other subsets already seen (→ no intersection of the basis)

2.11 Dual spaces and reciprocal products

The dual space V^* of V is the **space of linear functions** on V:

$$V^* = \{\boldsymbol{f} : V \to \mathbb{R} | \boldsymbol{f}\ linear\} \qquad \dim(V) = n \iff \dim(V^*) = n$$

$$\forall \text{ basis } \{\boldsymbol{e}_1, ..., \boldsymbol{e}_n\} \text{ of } V \quad \exists! \textbf{ dual basis } \{\boldsymbol{e}_1^*, ..., \boldsymbol{e}_n^*\} \text{ defined by } \boldsymbol{e}_j^*(\boldsymbol{e}_i) = \delta_{i,j}$$

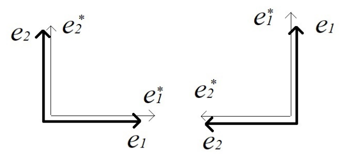
An example is the **space of forces** acting on a particle, that is the **dual** to the space **of** particle **velocities**:

$$\vec{f}(\vec{v}) = \vec{f} \cdot \vec{v}$$

The wrenches, $se^*(3)$, are dual to the twists, $se(3)$, and for them a specific scalar product called **reciprocal product** is defined: this application of a wrench on a twist **measures the power** exerted by the system of forces for the instantaneous motion:

$$\boxed{\mathcal{P}_o w = \zeta(\xi) = \zeta \circ \xi = \zeta \circ \xi = \vec{f} \cdot \vec{v_O} + \vec{m_O} + \vec{\omega}}$$

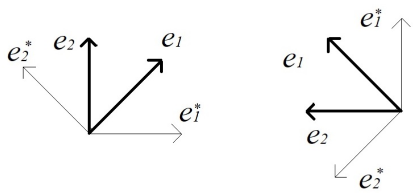
Consider a $V = \mathbb{R}$ with a orthogonal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$:



As seen in the Figure, the dual space basis $\{\boldsymbol{e}_1^*, \boldsymbol{e}_2^*\}$ of $V*$ is **identical** for the orthogonal basis, invariant to any change of base with another orthogonal basis:

$$\boldsymbol{e}_i^*(\boldsymbol{e}_j) = \delta_{i,j}$$

Now consider again $V = \mathbb{R}$ with a **non orthogonal** basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$:



In this case, in order to respect $\boldsymbol{e}_i^*(\boldsymbol{e}_j) = \delta_{i,j}$, the basis are not identical, but **isomorphic** and invariant to change of base (maintaining the angle α between $\boldsymbol{e}_1, \boldsymbol{e}_2$)

Some notes:

- It is important to notice that $\boxed{(V^*)^* = V}$, and this identity is completely canonical and true.
- It is important also to note that, for the Plücker twist basis $\{\rho_{Ox}\rho_{Oy}\rho_{Oz}\tau_x\tau_y\tau_z\}$ the dual wrench basis is $\{\mu_x\mu_y\mu_z\varphi_{Ox}\varphi_{Oy}\varphi_{Oz}\}$
- When dual bases are used – interpreting as column coordinate vectors $\zeta \circ \xi = \zeta^T \xi \rightarrow$ hence, the notation $\zeta \cdot \xi$ is also used.

Why the reciprocal product is important? Because it gives us the $\mathcal{P}$: you can't measure the magnitude in an intrinsic way, you need the dual $\rightarrow$ you can't identify the twists, the motions, without the dual wrenches, the forces. To understand it better, think about the following metaphor: ξ are the elements of a supermarket, and ζ are the associated prices. When you have to compute the total value of your merchandise, you need the reciprocal product in order to get the "total magnitude":

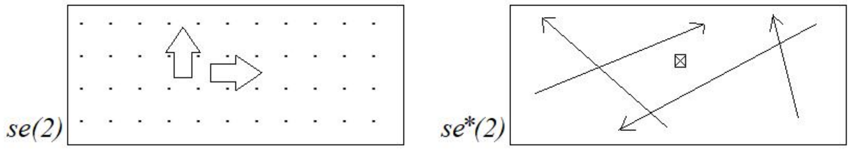
$$\text{Basis: } \begin{Bmatrix} \text{Potatoes} \\ \text{Oranges} \\ \text{Beer} \\ \vdots \end{Bmatrix} \qquad \text{Total value: } \begin{bmatrix} 0.3€/\text{Pot.} & 0.2€/\text{Or.} & 1.6€/\text{Beer} & \cdots \end{bmatrix} \cdot \begin{bmatrix} 100 \text{ Potatoes} \\ 150 \text{ Oranges} \\ 77 \text{ Beer} \\ \vdots \end{bmatrix}$$

Consider the following reciprocal product between two screws applied on the same point O:

$$\zeta_O(\xi_O) = \zeta_O \circ \xi_O = (f\hat{z}, hf\hat{z}) \circ (\omega\hat{z}, h\omega\hat{z}) = 2\omega f h$$

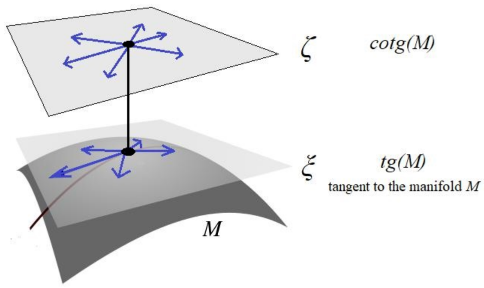
For rotations ($h = 0$) it does not give us any information about rotation. What? So the power is null?

Considering a 2-dimensional space, the **planar motions** $se(2)$, the rotations are points, while in the dual space $se^*(2)$ they are lines (you can think of lines as a function of the point):



2.12 Reciprocity and annihilators

Considering the group of rigid motions of Euclidean space, a screw is an element of the Lie algebra of this group (see later for advanced screw theory w.r.t differential geometry). Because of the properties of the Euclidean group (see Section 4.5), we are allowed to define Lie algebras as tangent bundled to the group, so then the twists and the wrenches will "live" on hyperplanes tangent to the manifold M of rigid motions of Euclidean space:



When a dual vector maps a vector into zero, the two are said to be orthogonal.

$$e.g.: \quad \mathcal{P} = \zeta(\xi) = \zeta \circ \xi = 0 \Rightarrow \zeta \perp \xi \quad \leftarrow \textbf{reciprocal} \text{ (null power)}$$

Considering a vector space $U \subset V$, $U^\perp \subset V^*$ and it will be:

$$U^\perp = \langle \zeta \in V^* \mid \zeta \cdot \xi \quad \forall \xi \in U \rangle \leftarrow \textbf{orthogonal annihilator} \text{ of } U$$

$U^\perp$ is a subspace.

In Euclidean spaces V and $V*$ are identified, and we have $U \oplus U^\perp = V \rightarrow$ If $U \subset se(3)$ then $\dim U + \dim U^\perp = 6$.

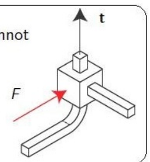
For U being a twist subspace, $U^\perp$ is a wrench subspace composed of all wrenches that exert no power on any motion in $U \rightarrow U^\perp$ is a **constraint system** allowing only motions in U.

Let's see now the **geometric conditions** for the following screws to be reciprocal:

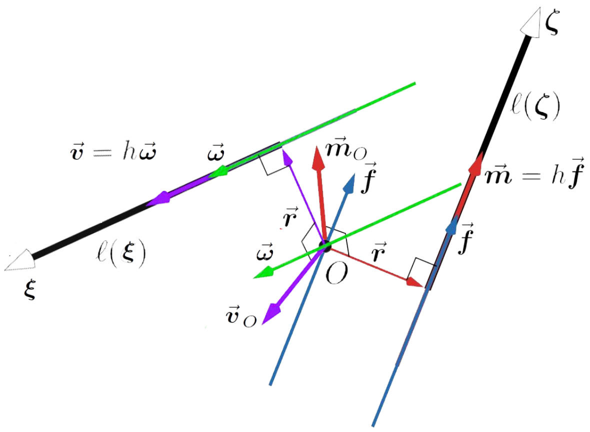
wrench	twist	reciprocity condition
Force $F = (\mathbf{s}, \mathbf{r}_1 \times \mathbf{s})$	Rotation $(\mathbf{u}, \mathbf{r}_2 \times \mathbf{u})$	coplanar axes
Force $F = (\mathbf{s}, \mathbf{r}_1 \times \mathbf{s})$	Translation $(0, \mathbf{t})$	$\mathbf{s} \perp \mathbf{t}$
Moment $M = (\mathbf{0}, \mathbf{m})$	Rotation $(\mathbf{u}, \mathbf{r}_2 \times \mathbf{u})$	$\mathbf{m} \perp \mathbf{u}$
Moment $M = (\mathbf{0}, \mathbf{m})$	Translation $(0, \mathbf{t})$	always

e.g. A pure force perpendicular to **t** cannot produce any translation along **t**

$\longrightarrow$ They are reciprocal, so the power exerted by the force for the instantaneous motion is null



Let's compute now the general case. Consider a twist and a wrench applied respectively on P_ξ and P_ζ (remind: in the following Figure $\vec{v_O}$ and $\vec{m_O}$ are **going out from the page**, since it is considered $h > 0$):



$$\zeta \circ \xi = (\vec{f}, \vec{m_O}) \circ (\vec{\omega}, \vec{v_O}) = (\vec{f}, h_\zeta \vec{f} + \vec{r_\zeta} \times \vec{f}) \circ (\vec{\omega}, h_\xi \vec{\omega} + \vec{r_\xi} \times \vec{\omega}) =$$
$$= \vec{f} \cdot \vec{v_O} + \vec{m_O} \cdot \vec{\omega} = \vec{f} \cdot h_\xi \vec{\omega} + \vec{f} \cdot \vec{r_\xi} \times \vec{\omega} + h_\zeta \vec{f} \cdot \vec{\omega} + \vec{r_\zeta} \times \vec{f} \cdot \vec{\omega} = 0$$

Now, considering θ the angle between the two 3D vectors:

$$\rightarrow h_\xi f \omega \cos(\theta) + \vec{f} \cdot \vec{r_\xi} \times \vec{\omega} + h_\zeta f \omega \cos(\theta) + \vec{\omega} \cdot \vec{r_\zeta} \times \vec{f} \rightarrow (\text{using } \vec{a} \cdot \vec{b} \times \vec{c} \rightarrow \vec{b} \cdot \vec{c} \times \vec{a})$$

$$\rightarrow \boxed{\zeta \circ \xi = (h_\xi + h_\zeta) f \omega \cos(\theta) + (\vec{r_\xi} + \vec{r_\zeta}) \cdot \vec{f} \times \vec{\omega} = 0}$$

1. If they pass through the **same point** ($P_\xi = P_\zeta$):

$$(\vec{r_\zeta} + \vec{r_\zeta} = \vec{r}) \rightarrow (h_\xi + h_\zeta) f \omega \cos(\theta) + \vec{r} \cdot \vec{f} \times \vec{\omega} = 0 \iff \begin{cases} \theta = 0 \vee h_\xi = -h_\zeta \\ \vec{r} = \mathbf{0} \vee \vec{f} /\!/ \vec{\omega} \end{cases}$$

   then:

   $\rightarrow \xi \perp \zeta$ and passing both through O

   $\rightarrow \xi$ and ζ passing both through O with opposite pitches $h, -h$

   $\rightarrow \xi /\!/ \zeta$ passing both through P with opposite pitches $h, -h$

2. If they are **perpendicular** ($\theta = 0$):

$$\zeta \circ \xi = 0 + (\vec{r_\zeta} + \vec{r_\zeta}) \cdot \vec{f} \times \vec{\omega} = 0 \iff \begin{cases} \vec{r_\xi} = -\vec{r_\zeta} \end{cases}$$

   then:

   $\rightarrow \xi \perp \zeta$ and passing both through O (id. Case 1)

   $\rightarrow \xi \perp \zeta$ and passing through opposite points (P_x, P_y) and $(-P_x, -P_y)$

3. If they are **parallel** ($\theta = 1, \vec{f} \times \vec{\omega} = \vec{0}$)

$$\zeta \circ \xi = (h_\xi + h_\zeta) f \omega = 0 \iff \begin{cases} h_\xi = -h_\zeta \end{cases}$$

   then:

   $\rightarrow \xi /\!/ \zeta$ with opposite pitches $h, -h$

   (then, for the one in Case 1, the "passing through the same point P" condition is not necessary)

- consider now the case with infinite pitches screws:

4. If one is a **translation** $\xi = \tau$ ($h_\tau = \infty$):

$$\zeta \circ \tau = (\vec{f}, \vec{m_O}) \circ (\vec{0}, \vec{v_O}) = f v_O \cos(\theta) = 0 \qquad \forall \zeta \perp \tau$$

5. If one is a **couple** $\zeta = \mu$ ($h_\mu = \infty$):

$$\mu \circ \xi = (\vec{0}, \vec{m_O}) \circ (\vec{\omega}, \vec{v_O}) = m_O \omega \cos(\theta) = 0 \qquad \forall \mu \perp \xi$$

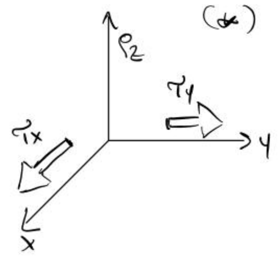
6. If one is a **translation** $\xi = \tau$ ($h_\xi = \infty$) and the other one is a **couple** $\zeta = \mu$ ($h_\zeta = \infty$):

$$\mu \circ \tau = (\vec{0}, \vec{m_O}) \circ (\vec{0}, \vec{v_O}) = 0 \qquad \forall \mu, \tau$$

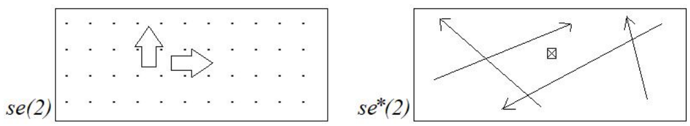
Let's now make some practical examples:

Ex. Describe $U^\perp$ when U is the **planar motions**:

The planar motions group $se(2)$ is composed by the translations $\perp z$ and by the rotations $\perp$ to the plane O_{xy} (//z). A basis of $se(2)$ is $\{\tau_x, \tau_y, \rho_{Oz}\}$ (2 dim space):



It spans two types of planar twists: either rotation //z axis or translation $\perp$ to plane O_{xy}. As we saw, looking from above (the $(*)$ in the picture) $se(2)$ rotations "look like points", and the reciprocal wrenches are lines on the plane:



Now we know why. In fact, if we search the span of the reciprocal space $se^\perp(2)$:

$$\dim(se^\perp(2)) = 6 - \dim(se(2)) = 3 \qquad \rightarrow \qquad Span(se^\perp(2)) = \{?, ?, ?\}$$

We have to find three wrenches ζ s.t.:

$$\begin{cases} \zeta \circ \tau_x = 0 \\ \zeta \circ \tau_y = 0 \\ \zeta \circ \rho_{Oz} = 0 \end{cases}$$

The **procedure** is the following one:

1. Find all the linearly independent ∞-pitch screw of $se^\perp(2)$ (**couples** μ)
$\rightarrow$ Can be done without computing any equation, just searching for the one $\perp$ to the non∞-pitch screws of the basis (normally, if we have a basis with only τ and ρ, this means $\perp$ to all the ρ)

2. Then find all the linearly independent 0-pitch screw of $se^\perp(2)$ (**pure forces** φ)
$\rightarrow$ Normally, you just have to find a force coplanar to all the ρ and $\perp$ τ. If necessary, compute it from the equations shown above

So, applying the procedure:

1. The moments $\perp \rho_z$ are the ones on the O_{xy} plane
   $\rightarrow$ we take two independent ones: μ_x and μ_y
2. The force that is coplanar to the ρ_z and $\perp \tau_{\in Oxy}$ is only φ_{Oz} (passing through O)

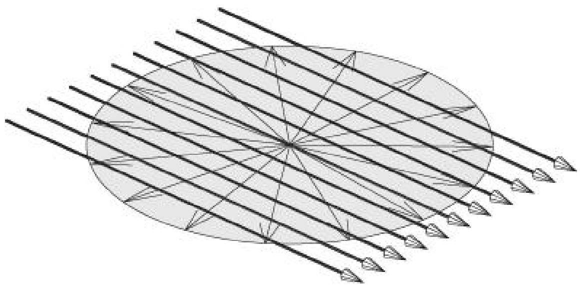
Then, we have the following possible base:

$$basis(se^{\perp}(2)) = \{\mu_x, \mu_y, \varphi_{Oz}\}$$

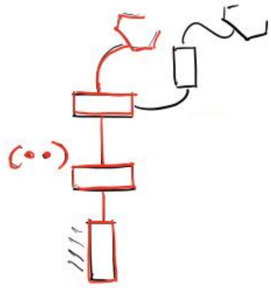
$\rightarrow Span\{\mu_x, \mu_y, \varphi_{Oz}\}$ = are the vertical **constraining** force along z and the constraining moments that constraints the rotations along any axis different from z ($\rightarrow$ this keeps the body on the plane O_{xy} defining the motion as "planar").

Be careful! The forces are not exercising **any power** on the body (reciprocity = null power), that is why they are constraints and not active forces!

Ex. Describe $U^{\perp}$ when U is the system spanned by the rotations in a plane:



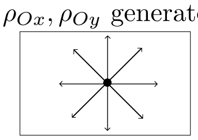
This system can be spanned by three rotations $\{\rho_i \in se(2)\}$, two with parallel axis and two with intersecting axis. A manipulator with this kind of rotations could be one of the two here:



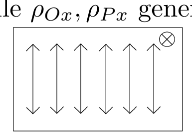
Why we choose these characteristics? Let's try to see if they are a sound basis for U:

$$basis(U) =? \{\rho_{Ox}, \rho_{Oy}, \rho_{Px}\}$$

ρ_{Ox}, ρ_{Oy} generate



while ρ_{Ox}, ρ_{Px} generate



So, another possible basis for the second generated space would be ρ_{Ox}, ρ_{Px} would be ρ_{Ox}, τ_z. Then, another possible basis for U, if $\{\rho_{Ox}, \rho_{Oy}, \rho_{Px}\}$ is a basis for it, would be $\{\rho_{Ox}, \rho_{Oy}, \tau_z\}$, but this for sure is a basis for it (they are all linearly independent):

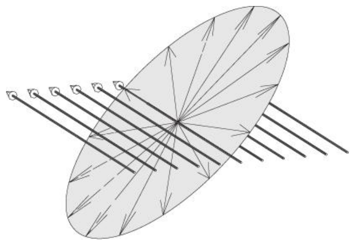
$$basis(U) = \{\rho_{Ox}, \rho_{Oy}, \tau_z\} = \{\rho_{Ox}, \rho_{Oy}, \rho_{Px}\} \qquad \dim(U) = 3$$

Then, $\dim(U^{\perp}) = 6 - 3$, then we have two find 3 linearly independent ζ:

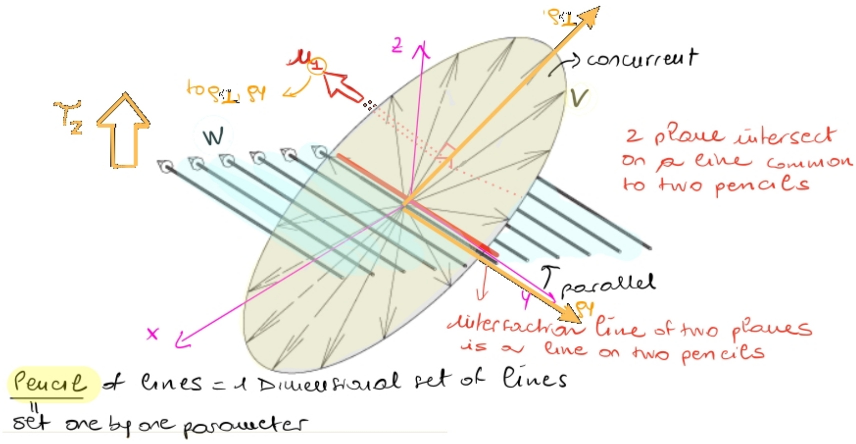
- Starting from the μ, the only $\perp \rho_{\in Oxy}$ is μ_z
- Regarding them φ, the ones $\in O_{xy}$ plane are the only one $\perp$ to the translations τ_z and coplanar to $\rho_{\in Oxy}$

$\Rightarrow$ $base(U^{\perp}) = \{\varphi_{Ox}, \varphi_{Oy}, \mu_z\}$

Ex. Describe $U^{\perp}$ when U is the system spanned by the rotations shown:



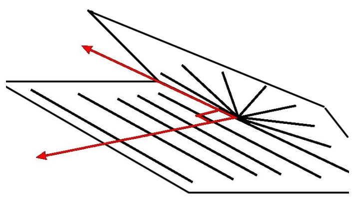
One basis can be found following some basic geometrical rules about pencils and their intersection:



In fact, every pencil is a mono-dimensional projective space, then is projection of a 2 dimensional vector space $\rightarrow$ every pencil's vector space is a dim = 2 vector space. In this case, we have also an intersection between the first pencil (the one with parallel rotations) and the second one (the one with concurrent rotations), then the intersection is a projective point, "a monodimensional line in vector spaces' world" $\rightarrow$ the intersection has dimension one, then the total dimension is 3:

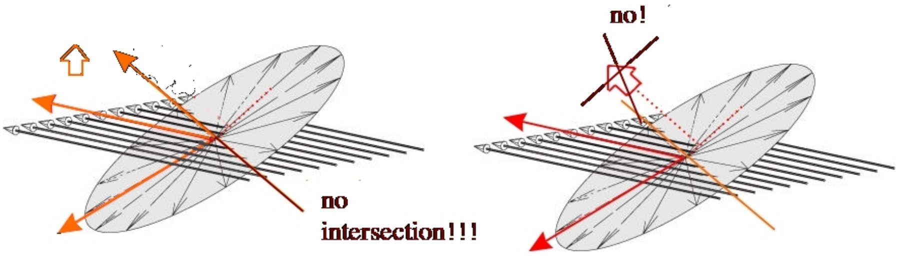
$$basis(U) = \{\rho_{Oy}, \rho_{\perp}, \tau_z\} \qquad \rightarrow \qquad basis(U^{\perp}) = \{\mu_1, ?, ?\}$$

The μ as seen in the picture is easy to find. Regarding the φ (remind: they have to be **orthogonal to τ!!**):



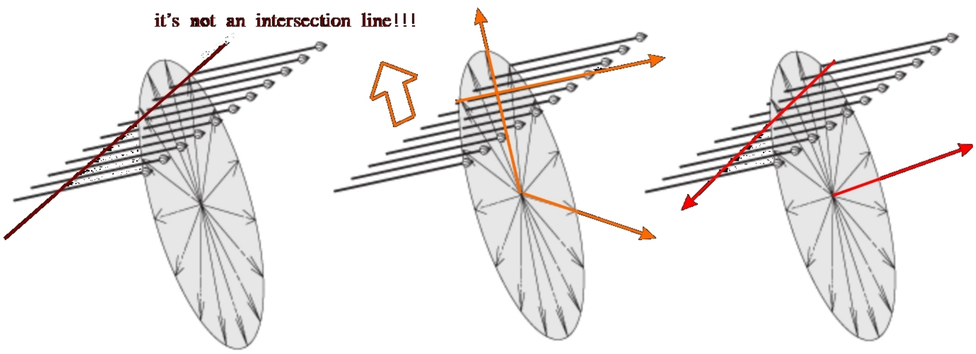
$$basis(U^{\perp}) = \{\mu_1, \varphi_{Ox}, \varphi_{Oy}\}$$

Ex. Describe $U^{\perp}$ when U is the system spanned by the rotations shown:



In this case there is no intersection!! The intersection line is not a line common to the two pencils, then the rotation of the **concurrent** ones on the plane of the **parallel** ones generates the span of all the rotations in the plane $\rightarrow \dim U = 4, \dim U^{\perp} = 2$

Ex. Describe $U^{\perp}$ when U is the system spanned by the rotations shown:



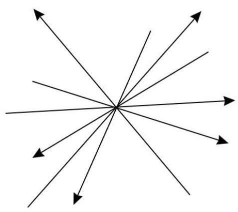
In this case, again there is no intersection $\rightarrow \dim U = 4, \dim U^{\perp} = 2$

2.13 Screw Systems

The **projective space** underlying a twist or wrench subspace is called a **screw system**. An n-system underlies an n-dimensional subspace.

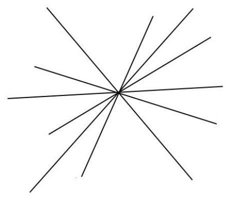
- $\rightarrow$ Two screw systems are **reciprocal** when any wrench acting on a screw in one system exerts no power on any twist on a screw in the other system.
  - $\rightarrow$ A screw system can be **self-**reciprocal.
- $\rightarrow$ Two systems are classified as **equivalent** when there is a rigid body displacement that can make them coincide.
- $\rightarrow$ For classification purposes it is sufficient to consider only systems underlying subspaces of dimension 2 or 3. (If higher, study the reciprocal.)

Consider $Span(\varphi_1, \varphi 2) \subset se^*(3)$ with $\varphi_1, \varphi 2$ concurrent:



$Span(\varphi_1, \varphi 2)$ = "forces with line of action in the plane through the point"

This vector space has dimension dim = 2. The underneath **projective space** is the space of **lines** of dimension $n-1$ (in this case dim = 1, i.e. the parameter is just one angle):

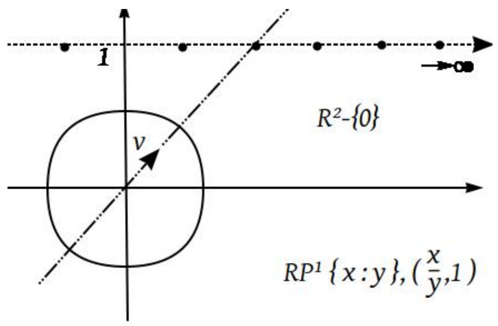


These are the lines through a point (this is RP^1, the "Real Projective space 1"), they are a **line pencil** of dimension 1. But we consider the vector space, not the linear space → vectorially, it has dimension 2, it's a **2-system**. what the heck???

All the screw systems are projective spaces, then let's analyze what does it mean to **projectivize** something like:

$$\mathbb{R}^2 \qquad \dim(\mathbb{R}^2) = 2 \qquad P(\mathbb{R}^2) \simeq RP^1$$

Projectivize $se(3)$ is removing the 0 ($\mathbb{R}^2/\{0\}$) and **parametrize** the others for example, for RP^1, that is $P(\mathbb{R}^2)$, we have:



For a vector $\vec{v} \in \mathbb{R}^2$, we have a line passing through the origin and intersecting a line in $y = 1$ (see figure up here). Then, this line will be uniquely identified by an angle (two same lines to be fair), and on the line $(x, 1)$ there will be an unique point for each line → that's the projective space. In the projective space (in this case just a line), the two dimensions are "wrapped": (x/y) is the only coordinate.

For screws is different, because we have also the h, so:

$$\dim(se(2)) = 3 \quad \leftarrow (\mathbb{E}^2, h) \qquad P(se(3)) \simeq RP^2 \quad \leftarrow (P(\mathbb{E}^2), h) = (RP^1, h)$$

on this part up here i'm not sure at all

The Gibson-Hunt Classification Screw systems are labelled by:

2 or 3	Dimension
I or *II*	Contains or not screws of more than one finite pitch
$A, ..., D$	The number from 0 to 3 of the independent 1-screws in the system
angle, pitch	Additional parameters where needed

From *Geometric Fundamentals of Robotics* – J.M. Selig, Springer we have these tables, were screws are generally indicated with $\boldsymbol{s}_i$ and their pitches with p:

TABLE 8.1. The 2-Systems

Completion Group	Gibson–Hunt Type	Normal Form	Isotropy Group
$\mathbb{R}^2$	IIC	$\mathbf{s}_1 = (0,0,0,1,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$	$E(2) \times \mathbb{R}$
$SO(2) \times \mathbb{R}$	IB^0	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,0,0,1,0,0)^T$	$O(2) \times \mathbb{R}$
$SO(3)$	IIA $(p = 0)$	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,0,0)^T$	$O(2)$
$H_p \ltimes \mathbb{R}^2$	IIB $(p \neq 0)$	$\mathbf{s}_1 = (1,0,0,p,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R}^2$
$SE(2)$	IIB $(p = 0)$	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R}^2$
$SE(2) \times \mathbb{R}$	IB $(p \neq 0)$	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,0,0,1,p,0)^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R}$
$SE(3)$	IIA $(p \neq 0)$	$\mathbf{s}_1 = (1,0,0,p,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p,0)^T$	$O(2)$
$SE(3)$	IA $(p_a \neq p_b)$	$\mathbf{s}_1 = (1,0,0,p_a,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p_b,0)^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

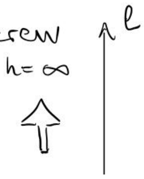
TABLE 8.2. The 3-Systems

Completion Group	Gibson–Hunt Type	Normal Form	Isotropy Group
$\mathbb{R}^3$	IID	$\mathbf{s}_1 = (0,0,0,1,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$ $\mathbf{s}_3 = (0,0,0,0,0,1)^T$	$SE(3)$
$SO(3)$	IIA $(p = 0)$	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,0,0)^T$ $\mathbf{s}_3 = (0,0,1,0,0,0)^T$	$SO(3)$
$SE(2)$	IIC $(p = 0)$	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$ $\mathbf{s}_3 = (0,0,0,0,0,1)^T$	$E(2) \times \mathbb{R}$
$H_p \ltimes \mathbb{R}^2$	IIC $(p \neq 0)$	$\mathbf{s}_1 = (1,0,0,p,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$ $\mathbf{s}_3 = (0,0,0,0,0,1)^T$	$E(2) \times \mathbb{R}$
$SE(2) \times \mathbb{R}$	IC $(p \neq 0)$	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$ $\mathbf{s}_3 = (0,0,0,1,0,p)^T$	$\mathbb{R}^2$
$SE(2) \times \mathbb{R}$	IC^0	$\mathbf{s}_1 = (1,0,0,0,0,0)^T$ $\mathbf{s}_2 = (0,0,0,0,1,0)^T$ $\mathbf{s}_3 = (0,0,0,1,0,0)^T$	$\mathbb{Z}_2 \times \mathbb{R}^2$
$SE(3)$	IIB	$\mathbf{s}_1 = (1,0,0,p,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p,0)^T$ $\mathbf{s}_3 = (0,0,0,0,0,1)^T$	$E(2)$
$SE(3)$	IB_0 $(p_a \neq p_b)$	$\mathbf{s}_1 = (1,0,0,p_a,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p_a,0)^T$ $\mathbf{s}_3 = (0,0,0,1,0,p_b)^T$	$\mathbb{Z}_2$
$SE(3)$	IB_3 $(p_a \neq p_b)$	$\mathbf{s}_1 = (1,0,0,p_a,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p_b,0)^T$ $\mathbf{s}_3 = (0,0,0,0,0,1)^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{R}^2$
$SE(3)$	IIA $(p \neq 0)$	$\mathbf{s}_1 = (1,0,0,p,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p,0)^T$ $\mathbf{s}_3 = (0,0,1,0,0,p)^T$	$SO(3)$
$SE(3)$	IA_2 $(p_a \neq p_b)$	$\mathbf{s}_1 = (1,0,0,p_a,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p_b,0)^T$ $\mathbf{s}_3 = (0,0,1,0,0,p_b)^T$	$O(2)$
$SE(3)$	IA_1, $(p_a \neq p_b \neq p_c \neq p_a)$	$\mathbf{s}_1 = (1,0,0,p_a,0,0)^T$ $\mathbf{s}_2 = (0,1,0,0,p_b,0)^T$ $\mathbf{s}_3 = (0,0,1,0,0,p_c)^T$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

Let's see some of these systems:

– 1 Systems 1 screw

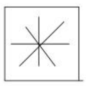
described by
- an axis
- a pitch



span(ξ)

described by $h(\xi)$

– 2 systems 2 screws

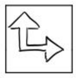


m=RL
↑
dimension of vector space

$L = A, B, C, D$ (0, 1, 2, 3)
how many independent $h=\infty$ there are

$R = I, II$
I: many finite pitches
II: only one finite pitch

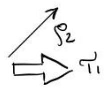
2.1)



CLASS 2-II-C

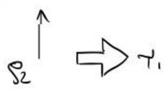
only one finite pitch
2 independent $h=\infty$ screw

2.2)

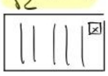


span(τ_1, ρ_2)

2.2.1.)

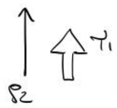


$\rho_2 \perp \tau_1$



2-II-B only one finite pitch
one $h=\infty$

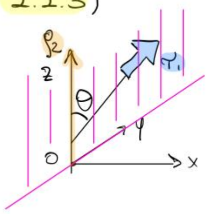
2.2.2)



$\rho_2 \parallel \tau_1$ ∅ case → angle is ∅

2-I-B many finite pitch
one $h=\infty$

2.2.3)



$$a_1 \,|\, \tau_1 = (\vec{0}, \cos\theta\,\vec{\imath} + \sin\theta\,\vec{k})$$
$$a_2 \,|\, \rho_2 = (\vec{k}, \vec{0})$$

$$\xi = (a_2\vec{k},\ a_1\cos\theta\,\vec{\imath} + a_2\sin\theta\,\vec{k})$$

$$h(\xi) = \frac{a_1}{a_2}\sin\theta$$ the screw can have any pitch

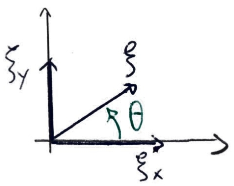
$\sin\theta \neq 0$ always because we have already considered the case $\rho \parallel \tau$

$$r_\perp = \underset{\text{something}}{(*)}\,\vec{\jmath}$$

$$r_\perp = \frac{a_1}{a_2}\cos\theta\,\vec{\jmath}$$

Now let's consider a more general case:

CONSIDER:



$\xi_x = (\hat{x}, h_x \hat{x})$ $\quad h_x, h_y < \infty$

$\xi_y = (\hat{y}, h_y \hat{y})$ $\quad h_x > h_y \rightarrow$ NOT CONSIDERING $h_x = h_y$

$\mathrm{SPAN}(\xi_x, \xi_y) \ni \xi \rightarrow$ A LIN. COMB. OF THEM

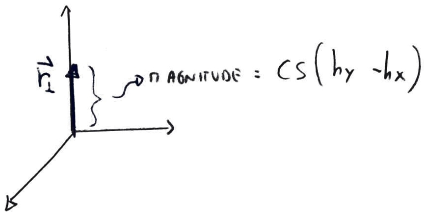
$$c \cdot \xi_x + s \cdot \xi_y = (c\hat{x} + s\hat{y}, c h_x \hat{x} + s h_y \hat{y}) \rightarrow \text{GENERIC ELEMENT}$$

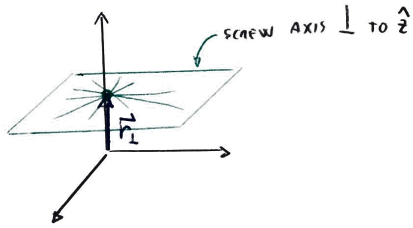
PITCH AND AXIS?

$$h = \frac{c^2 h_x + s^2 h_y}{c^2 + s^2}$$ $\rightarrow$ CONSIDER THE GENERIC UNIT VECTOR ELEMENT IF $\rightarrow$ CONSTRAINT: $c^2 + s^2 = 1$ $\quad (\cos^2\theta + \mathrm{sen}^2\theta = 1)$

$\xi_{unit} \rightarrow h = c^2 h_x + s^2 h_y \longrightarrow$ THIS CANNOT BE $\infty \rightarrow$ THE SET WON'T CONTAIN PURE TRANSLATIONS

$$\vec{r}_\perp = \frac{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ c & s & \emptyset \\ ch_x & sh_y & \emptyset \end{vmatrix}}{1} = (\emptyset, \emptyset, cs h_y - cs h_x) = cs(h_y - h_x)\hat{z}$$





$c^2 + s^2 = \cos^2\theta + \mathrm{sen}^2\theta = 1 \Rightarrow cs = \frac{1}{2}\varsigma$ $\quad$ WITH $\varsigma = \frac{1}{2}\mathrm{sen}(2\theta)$

$$h = c^2 h_x + (1 - c^2) h_y = \underbrace{h_y + c^2(h_x - h_y)}$$

$$h_y < h < h_x$$

(BECAUSE $\emptyset < c^2 < 1$)

$$\vec{r}_\perp = \underbrace{\frac{1}{2}\varsigma(h_y - h_x)}_{:= d}\hat{z}$$

$$\frac{h_y - h_x}{2} < d < \frac{h_x - h_y}{2}$$

(BECAUSE $-1 < \varsigma < 1$ AND $h_y - h_x < \emptyset$)

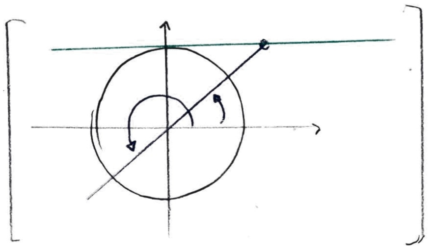
MIN: $d = d_{min} \leftrightarrow \theta = +45°$

MAX: $d = d_{max} \leftrightarrow \theta = -45°$

MIN: $h = h_y \leftarrow \theta = 90°$

MAX: $h = h_x \leftarrow \theta = \emptyset°$

IN RP^1, 2 ANGLES REPRESENT THE SAME POINT, THAT'S WHY WE HAVE $\mathrm{sen}(2\theta) = \varsigma$ $\Rightarrow$



$$d = \frac{1}{2} S' (h_y - h_x) = cs (h_y - h_z)$$

$$\begin{cases} x = Rc \\ y = Rs \end{cases}$$ POLAR COORDS. $\to$ $c = X/R$, $s = Y/R$

d IS ALONG $\vec{z}$ $\to$ IT IS "z"

$$\frac{z}{xy} = \frac{1}{R^2} (h_y - h_z) \Rightarrow \boxed{-\frac{x^2+y^2}{xy} \; \frac{z}{h_x - h_y} = \emptyset}$$

$R^2 = x^2 + y^2$

PARAMETER $(h_x - h_y)$

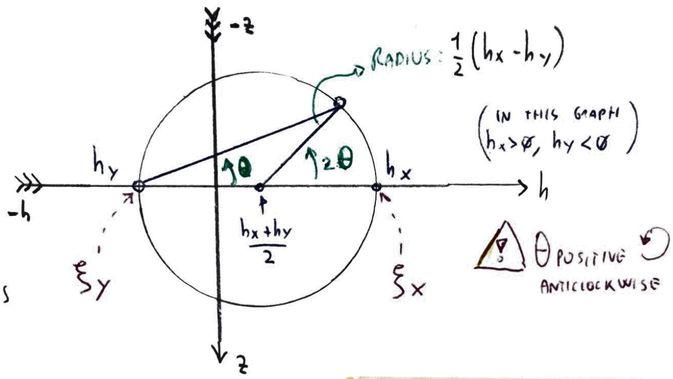
ELIMINATING CARTESIAN COORDINATES, THIS IS THE EQUATION OF A SURFACE ON THE SPACE THAT CONTAINS ALL THE SCREWS IN THE SPACE !

$\to$ CYLINDROID (OR "PLÜCKER CONOID"): THE PARAMETER IS $h_x - h_y$

GRAPHICAL REPRESENTATION:

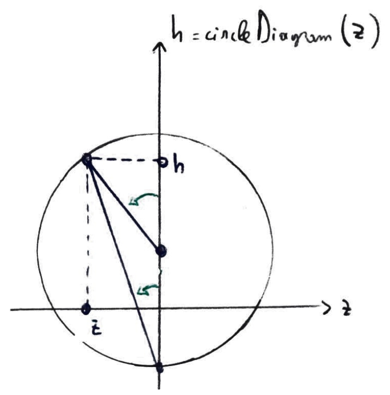
$$S = 2 \cdot c \cdot s \qquad C' = 2c^2 - 1 \to c^2 = \frac{1 + C'}{2} \qquad h = h_y + \left[\frac{1 + C'}{2}\right] (h_x - h_y)$$

$$\begin{cases} h = \dfrac{h_x + h_y}{2} + \dfrac{1}{2} C (h_x - h_y) \\ z = -\dfrac{1}{2} S' (h_x - h_y) \end{cases}$$



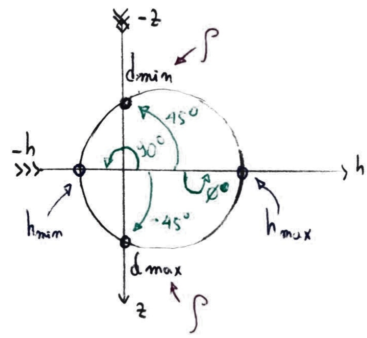
THIS IS THE CIRCLE DIAGRAM OF THE CYLINDROID, EVERY POINT CORRESPONDS TO A SCREW IN THE CYLINDROID.

THE z COORDINATE TELLS US HOW HIGH IS THAT SCREW, AND FROM THIS YOU OBTAIN THE PITCH: $\boxed{h = \text{Circle Diagram}(z)}$

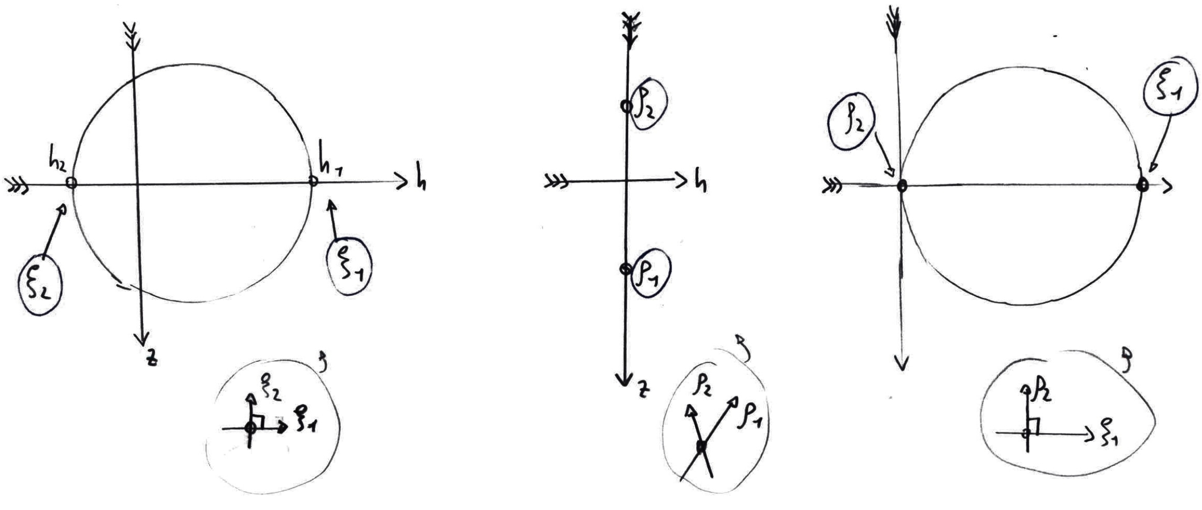


EVERY PITCH HAPPENS TWICE, EVERY ELEVATION HAPPENS TWICE, WITH THE EXCEPTION OF THE EXTREME PITCHES h_x, h_y AND THE EXTREME ELEVATIONS d_{max}, d_{min}

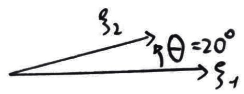
$\to$



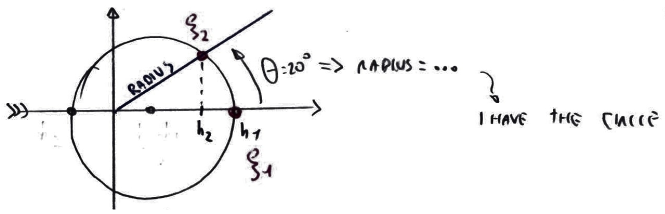
WE CAN THEN EXTRACT THE PARTICULAR CASES:



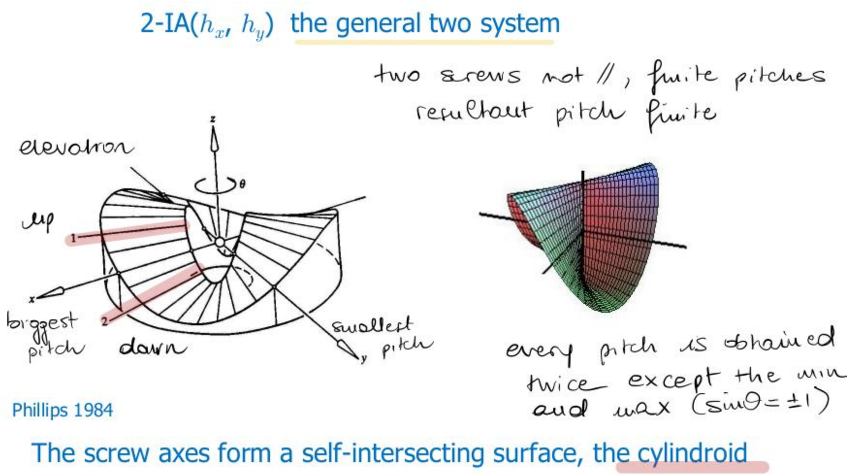
EXAMPLE: WE HAVE



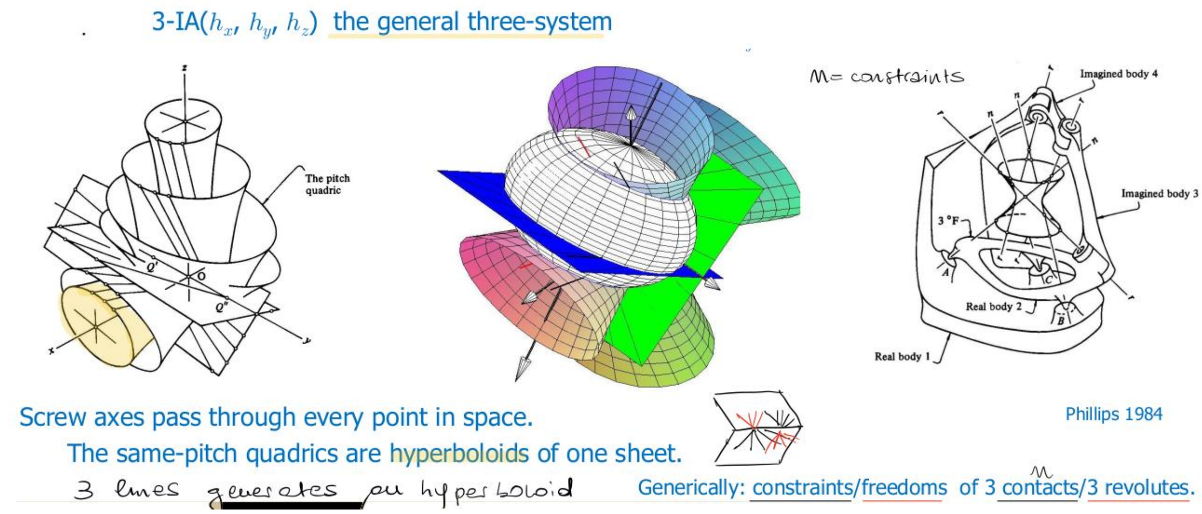
IF YOU HAVE 2 SCREWS, YOU JUST PLOT THEM IN THE PLOT AND YOU GENERATE THE CIRCLE BASED ON THE ANGLE AND h_1, h_2

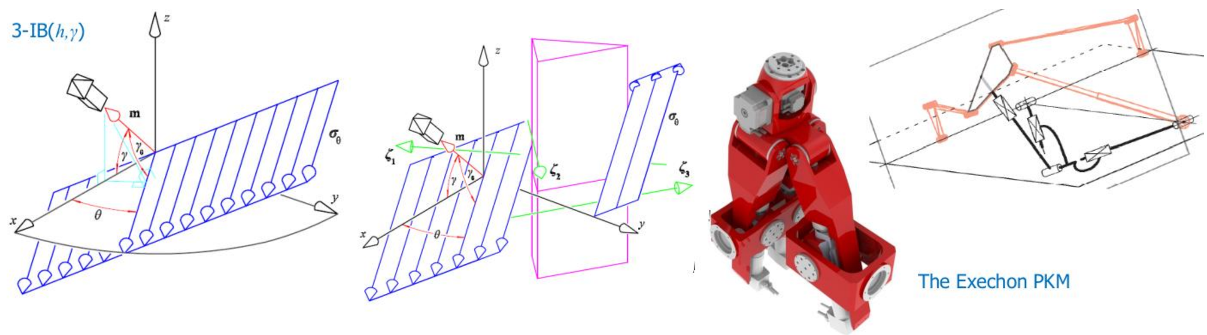


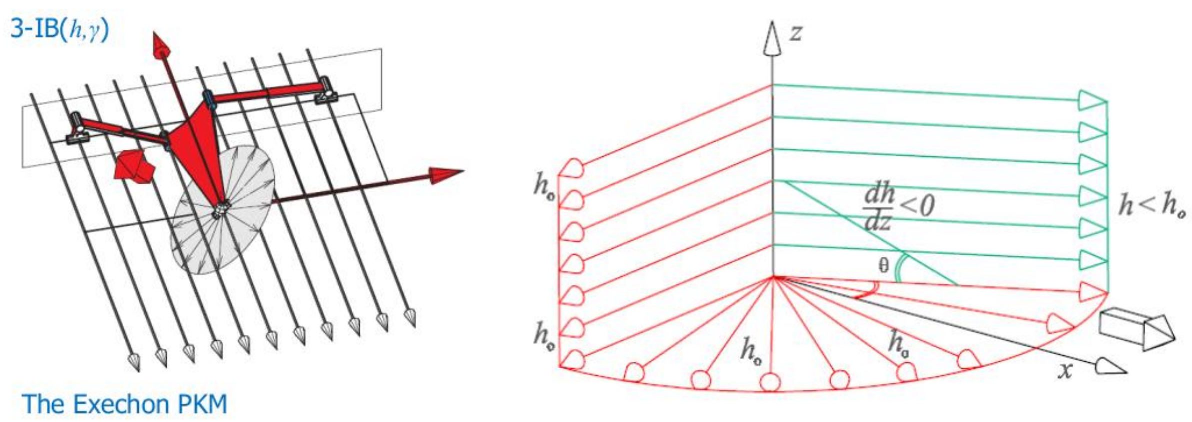
This is the 3D graphical representation:



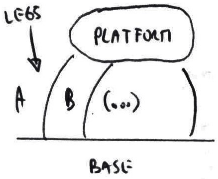
Other examples of screw systems are:







3 Constraint analysis



"PLATFORM FREEDOMS" DETECTION → "POSSIBLE END EFFECTOR <u>TWIST</u>" DETECTION

$$T = \underbrace{\bigcap_{\ell = A,B,\ldots}}_{\text{INTERSECT ALL THE LEGS}} T_{LEG_\ell}$$

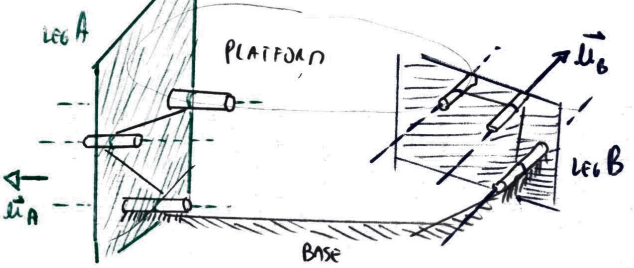
LEGS WITH CAPITAL LETTERS (← <u>NOTATION</u>)
DON'T USE NUMBERS, THEY ARE FOR THE SINGLE LEG'S JOINTS
↳ $\ell = \{A, B, C, D, \ldots\}$

(D-H (DENAVIT-HARTENBERG) NUMBERING WITH 1,2,3... FOR SERIAL CHAINS
A,B,C... NUMBERING FOR PARALLEL CHAINS.) ↲

ADDING NEW LEGS JUST ADDS CONSTRAINTS, NOT MOTIONS! WHILE ↳

$$T_{LEG} = \sum_{i=1,\ldots,m_{JOINTS}} \mathrm{SPAN}(\xi_i)$$

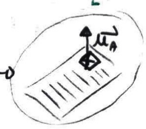
IT WOULD BE OBVIOUS TO COMPUTE SPAN FOR JOINT i, THEN SUM THEM IN T_{LEG} AND THEN INTERSECT THE VARIOUS T_{LEG_ℓ}
↙
IT'S COMPLICATED WHEN WE HAVE A LOT OF TWISTS ↳ WE USE <u>WRENCHES</u>



$\left(\begin{array}{l}3 \text{ PARALLEL } P \\ 1 \text{ DIRECTION: } \vec{u}_A\end{array}\right)$ ↓ PLANAR MOTIONS ON PLANES // TO THE SERIAL CHAIN ($\perp$ TO $\vec{u}_A$) → $\left[\begin{array}{l}\text{CAN ROTATE } \perp \vec{u}_A \\ \text{TRANSLATE } /\!/ \vec{u}_A\end{array}\right]$

$\left(\begin{array}{l}3 \text{ PARALLEL } P \\ 1 \text{ DIRECTION: } \vec{u}_B\end{array}\right)$ ↘ $\left[\begin{array}{l}\text{CAN ROTATE } \perp \vec{u}_B \\ \text{TRANSLATE } /\!/ \vec{u}_B\end{array}\right]$

$T_A = se(2)(\vec{u}_A)$ ~



$T_B = se(2)(\vec{u}_B)$

$T = T_A \cap T_B$

→ INTERSECTION → $\exists$ ROTATION $\perp \vec{u}_B$ <u>AND</u> $\perp \vec{u}_A$ ⇒ <u>NO</u> ROTATIONS IN T!

$$T^\perp = W \rightarrow T^\perp = \sum_\ell T^\perp_{LEG_\ell} = \sum_\ell W_{LEG_\ell} \Rightarrow T = \Big(\sum_\ell W_{LEG_\ell}\Big)^\perp$$

← EASIER!
STUDY CONSTRAINTS
↳ SEE <u>DESRO</u> COURSE

4 Advanced Screw Theory

4.1 Differential geometry (recall)

Groups, rings and fields:

- **Binary operation** $\star$ on set S
  let S be a non-empty set. A map
  $$\star : S \times S \to S, \quad (a, b) \mapsto a \star b$$
  is called a binary operation on S. So $\star$ takes 2 inputs a, b from S and produces a single output $a \star b \in S$. In this situation we may say that 'S is closed under $\star$'
  – unary operation: $\star : S \to S$ (e.g. $a \mapsto -a$)
  – $\star$ is commutative if, $\forall a, b, \in S, \quad a \star b = b \star a$
  – $\star$ is associative if, $\forall a, b, \in S, \quad a \star (b \star c) = (a \star b) \star c$
  – scalar product is NOT a binary operation on $\mathbb{R}^3$
- **Group** $(G; \star)$
  let G be a non-empty set and let $\star$ be a binary operation on G:
  $$\star : G \times G \to G, \quad (a, b) \mapsto a \star b$$
  Then $(G; \star)$ is a group if the following axioms are satisfied:
  – *associativity*: $a \star (b \star c) = (a \star b) \star c, \quad \forall a, b, c \in G \times G$
  – *identity element*: $\exists e \in G \; s.t. \; a \star e = e \star a = a, \quad \forall a \in G$
  – *inverses*: $\forall a \in G \;\; \exists a^{-1} \in G \;\; s.t. \;\; a \star a^-1 = a^{-1} \star a = e$
  $|G|$ = order of G = number of elements in it ($|G| = \infty$ for infinite groups)
  If in addition holds commutativity ($a \star b = b \star a, \;\; \forall a, b \in G$) then $(G; \star)$ is an **abelian** group
  e.g.: $(\mathbb{Z}; +), (\mathbb{Q}; +), (\mathbb{R}; +), (\mathbb{C}; +), (\mathbb{Q}\backslash\{0\}, \cdot), (\mathbb{R}\backslash\{0\}, \cdot), (\mathbb{C}\backslash\{0\}, \cdot)$ are abelian groups.
- **Ring** $(R; +, \cdot)$
  a structure $(R, +, \cdot)$ is a ring if R is a non-empty set and $+$ and $\cdot$ are binary operations:
  $$+ : G \times G \to G, \quad (a, b) \mapsto a + b \qquad \cdot : G \times G \to G, \quad (a, b) \mapsto a \cdot b$$
  such that
  – for $+$, $(R, +)$ is an abelian group
  – for $\cdot$ associativity holds: $\forall a, b, c \in R \;\; a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  – for $+, \cdot$ associativity holds $\forall a, b, c \in R, \;\; a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot b + b \cdot c$
  !!! We do NOT demand $\cdot$ in R commutative $\to$ 2 laws, one not follows from the other
  Assume $(R; +, \cdot)$ is a ring:
  – R is **commutative** if $\forall a, b \in R, \;\; a \cdot b = b \cdot a$
  – R is **with identity** if $\forall a \in R, \;\; 1 \cdot a = a$
  e.g.: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are commutative rings with identity (identity is number 1)
  e.g.: $\mathbb{N}$ is NOT a ring for the usual addition and multiplication(existence of additive inverses fails)

◦ Calculational rules for rings
  – if $(R; +, \cdot)$ commutative ring $(a, b, c \in R)$:
  $$a + b = a + c \Rightarrow b = c$$
  $$a + a = a \Rightarrow a = 0$$
  $$-(-a) = a$$

$0a = 0$
$-(ab) = (-a)b = a(-b)$

– if $(R; +, \cdot)$ commutative ring with identity:

$(-1)a = -a$
if $a \in R$ has a multiplicative identity $a^{-1} \Rightarrow [ab = 0 \Rightarrow b = 0]$

- **Field** $(F, +, \cdot)$
  a structure $(F, +, \cdot)$ is a field if
  – $(F, +)$ is an abelian group;
  – $(F \backslash \{0\}, \cdot)$ is an abelian group
  – the distributive laws hold
  e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields

◦ A commutative ring with identity $(R, +, \cdot)$ is *"almost a field" without inverses for* $\cdot$ operation
  $\rightarrow (R \backslash \{0\}, \cdot)$ is not a group
  e.g. $\mathbb{Z}$ fails to be a field

◦ In a commutative ring we call an element $a \neq 0$ a **zero divisor** if $\exists b \neq 0$ *s.t.* $a \cdot b = 0$

◦ A comm. ring with identity in which $1 \neq 0$ is an **integral domain** (ID) if has no zero divisors
  – cancellation property: in a ID $ab = ac \wedge a \neq 0 \Rightarrow b = c$
  – F is a field $\Rightarrow$ F is an integral domain
  – R is a field $\not\Leftarrow$ R is an integral domain (e.g. $\mathbb{Z}$)
  – F is a field $\Leftarrow$ F is an *finite* integral domain

- **Units** u
  – let R be a commutative ring with identity
  – $0 \neq u \in R$ is a unit if $\exists v \in R$ *s.t.* $uv = vu = 1$
  – we write $v = u^{-1} \rightarrow$ units are those non-zero elements which have multiplicative inverses
  – e.g. in a field F, every non-zero element is a unit. In $\mathbb{Z}$, the units are ± 1.

Relations and partitions:

- **Relation** $\sim$
  – fix a non-empty set Ω. A binary relation on Ω is (officially) a subset $\sim$ of $\Omega \times \Omega$
  – so the elements of $\sim$ are certain pairs $(a, b) \in \Omega \times \Omega$.
  – $\sim$ gives a true/false split: true if $(a, b) \in\sim$ and false if $(a, b) \notin\sim$
  – we use $a \sim b$ instead of $(a, b) \in\sim$

  We say $\sim$ is
  (R) reflexive if $a \sim a \quad \forall a \in \Omega$
  (S) symmetric if $a \sim b \Rightarrow b \sim a \quad \forall a, b \in \Omega$
  (T) transitive if $a \sim b \Rightarrow b \sim c \quad \forall a, b, c \in \Omega$

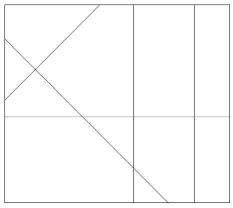
Examples

Ω	$\sim$	(R)	(S)	(T)
$\mathbb{R}$	$\leqslant$	Y	N	Y
$\mathbb{R}$	$<$	N	N	Y
lines in $\mathbb{R}^2$	"is parallel to"	Y	Y	Y
lines in $\mathbb{R}^2$	"is orthogonal to"	N	Y	N
$\mathbb{Z}$	$a \sim b$ iff $3 \mid (a-b)$	Y	Y	Y
$\mathbb{N} \setminus \{0\}$	"divides"	Y	N	Y
any Ω	$=$	Y	Y	Y
any Ω	$a \sim b \; \forall a, b$	Y	Y	Y

- **Equivalence relation** $\sim$
  - let Ω be a non-empty set and $\sim$ a relation on Ω
  - $\sim$ is an equivalence relation if satisfies (R), (S) and (T)
  - when $\sim$ is an equivalence relation have an **equivalence class** $[a]$ **of** a that is

$$[a] := \{b \in \Omega | a \sim b\}$$

- **Partition** $\mathcal{U}$
  - let $\sim$ equiv. rel. on a non-empty set Ω and $\mathcal{U} = \{U_i\}_{i \in I}$ a non-empty family of subsets of Ω
  - each element in Ω belongs to only one equivalence class
  - $\mathcal{U}$ is a partition of Ω if Ω is the disjoint union of the non-empty sets U_i (in Figure $I = [1, 11]$)



  - one-to-one correspondence between equivalence relations on Ω and partitions of Ω

◦ $(\Omega/\sim)$ is the **set of equiv. classes** for a equiv. rel. $\sim$ on a set ω (a.k.a "the quotient of Ω by $\sim$")
  - if we do not want to distinguish between elements which are related:
  - the map $x \mapsto [x]$ $(x \in \Omega)$ allows us to pass from Ω to $\Omega/\sim$

Algebras and vector spaces:

- **Algebraic structure** $(A, \{\omega\})$ on a set K
  - contains *operations* $\{\omega\}$ on K of finite arity (= number of arguments a function takes)
  - contains finite set of identities (*axioms* of the structure) that the operations must satisfy
  - term "**algebra**" is for specific structures (e.g. if A is a vector space over K, and K is a field)

◦ Algebraic structures can be
  Group-like (Group, Semigroup , Abelian group, Lie group ...),
  Ring-like (Ring, Semiring, Near-ring, Commutative ring, ID, Field ...),
  Lattice-like (Lattice, Semilattice, Total order, Heyting algebra, Boolean algebra ...),
  Module-like (Module, Group with operators, Vector space ...)
  Algebra-like (Algebra, Associative, Non-associative, Composition algebra, Lie algebra ...)

- **Module** $(M; +, \cdot)$ **over a ring** R
  - R is a ring and 1_R is its multiplicative identity
  - a **left** R**-module** ${}_RM$ is an abelian group $({}_RM, +)$ and an operation (SM) that is **not** a binary operation:

$$\begin{aligned}
&\cdot : R \times {}_RM \to {}_RM, \quad (r, x) \mapsto r \cdot x \\
&s.t. \quad \forall r, s \in R \quad x, y \in M \quad \text{we have:} \\
&- \quad r \cdot (x + y) = r \cdot x + r \cdot y \\
&- \quad (r + s) \cdot x = r \cdot x + s \cdot x \\
&- \quad (rs) \cdot x = r \cdot (s \cdot x) \\
&- \quad 1_R \cdot x = x
\end{aligned}$$

– for *right R-module* M_R the ring acts on the right, i.e. SM is defined as

$$\cdot : M_R \times R \to M_R, \quad (r, x) \mapsto x \cdot r$$

!!!: SM (not binary operation) is different from the ring multiplication operation (binary operation):

$\rightarrow$ the *ring multiplication* (e.g. rs) is a mult. of two elements of the set producing an element of the set (i.e. $M \times M \to M$ or $R \times R \to R$)

$\rightarrow$ the *scalar multiplication* (e.g $r \cdot x$ or $x \cdot r$) is a mult. of an element of the set with an element of the underlying ring producing an element of the set (i.e. $R \times {}_RM \to {}_RM$ or $M_R \times R \to M_R$)

- **Vector space** $(V; +, \cdot)$ over a field F

  set V with two operations (VA,SM) that satisfy eight axioms [2]

  – vector addition (VA) $+ : V \times V \to V,\ \ (\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} + \boldsymbol{w} \in V$

  – scalar multiplication (SM) $\cdot : F \times V \to V,\ \ (a, \boldsymbol{v}) \mapsto a\boldsymbol{v} \in V$

  !!!: Rings, fields, and IDs all require two binary operations. SM is not a binary operation

  $\rightarrow$ a vector space is a ${}_RM$ **module over a** ring that is a **field** F

- Notation (from now on):

  – scalar multiplication $R \times M \to M$: $a\boldsymbol{v}$

  – bilinear product $M \times M \to M$: $\boldsymbol{u} \cdot \boldsymbol{v}$

- Note: $\cdot$ operator "overloading". In definitions:

  – $(a, b) \mapsto a \cdot b$ with $a \in R$ and $b \in M \rightarrow$ scalar multiplication

  – $(a, b) \mapsto a \cdot b$ with $a \in M$ and $b \in M \rightarrow$ bilinear multiplication

  – $(a, b) \mapsto a \cdot b$ with $a \in R$ and $b \in R \rightarrow$ bilinear form (e.g. dot/inner/scalar product)

- **Algebra** $(A; +, \cdot)$ over a field F

  – is a vector space equipped with a bilinear product

  $$\cdot A \times A \to A, \ \ (\boldsymbol{u}, \boldsymbol{v}) \mapsto \boldsymbol{v} \cdot \boldsymbol{v}$$

  – multiplication may or may not be associative ($\rightarrow$ *associative* and nonassociative algebras)

  – is *unital* (or unitary) if it has an identity element with respect to the multiplication

  !!!: Algebras are *not to be confused* with vector spaces equipped with a bilinear form ($M \times M \to R$) like *inner product spaces*, as, for such a space, the result of a product is not in the space, but rather in the field of coefficients.

○ Sometimes we can consider the more general concept of an *algebra over a ring*, where a commutative unital ring R replaces the field K. The only part of the definition that changes is that A is assumed to be an R-module (instead of a vector space over K).

- **Associative algebra** $(A; +, \cdot)$ over a field F (F**-algebra**)

  – has bilinear addition and multip. (assumed associative) and SM by elements on a field F

  – addition + multiplication: A is a ring

  – addition + SM: A is a vector space over F

  e.g.: a ring of square matrices over a field F (elements $\in F$) with the usual matrix multiplication

[2] They are the same for the module, adapted to F:

Field addition (FA): $a + b$	Vector addition (VA): $\boldsymbol{u} + \boldsymbol{v}$		
Field multiplication (FM): $a \cdot b$	Scalar multiplication (SA): $a \cdot \boldsymbol{u}$		
For VA:	(1) associativity	(2) commutativity	(3) identity element (4) inverse elements
For SM:	(5) compatibility of SM with FM	(6) id. element of SM	(7) distributiv. of SM w.r.t. VA and (8) to FA

- **Lie algebra** $(\mathfrak{g}; [\cdot,\cdot])$ over a field F
  - is vector space $\mathfrak{g}$ over F with a binary operation called **Lie braket**:

$$[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}\quad\text{such that:}$$
$$-\ \begin{matrix}[ax+by,z]=a[x,z]+b[y,z]\\ [z,ax+by]=a[z,x]+b[z,y]\end{matrix}\qquad \forall a,b\in V\ \ \forall x,y,z\in\mathfrak{g}\ \ \text{(bilinearity)}$$
$$-\ [x,x]=0\qquad \forall a,b\in V\ \ \forall x,y,z\in\mathfrak{g}\ \ \text{(alternativity)}$$
$$-\ [x,[y,z]]+[z,[x,y]]+[y,[z,x]]=0\qquad \forall a,b\in V\ \ \forall x,y,z\in\mathfrak{g}\ \ \text{(Jacobi identity)}$$

  – bilinearity + alternativity imply anticommutativity ($[x,y]=-[y,x]\ \ \forall x,y\in\mathfrak{g}$

  – if the field's characteristic is not 2 then anticommutativity implies alternativity.

  – normally Lie algebra denoted by a lower-case fraktur letter $\mathfrak{g},\mathfrak{h},\mathfrak{b},\mathfrak{n}$ or with lower case letters e.g. if associated with a Lie group (for $SU(n)$, then the algebra is $\mathfrak{su}(n)$ or $su(n)$)

* Topology parenthesis:

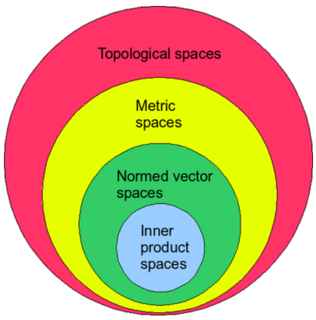
  * **Topological space**: set of points, along with a set of neighborhoods for each point, satisfying a set of axioms relating points and neighborhoods

  * **Metric space**: set together with a metric on the set. The metric is a function that defines a concept of distance between any two members of the set, which are usually called points. Formally:
  A metric space is an ordered pair (M,d) M is a set and d is a metric on M, i.e., a function

$$d\colon M\times M\to\mathbb{R}\qquad\text{such that:}$$
$$-\ d(x,y)=0\Leftrightarrow x=y\qquad\text{(identity of indiscernibles)}$$
$$-\ d(x,y)=d(y,x)\qquad\text{(symmetry)}$$
$$-\ d(x,z)\le d(x,y)+d(y,z)\qquad\text{(subadditivity or triangle inequality)}$$

  from these we obtain $d(x,y)\ge 0\ \ \forall\, x,y\in M$

- **Normed vector space** $(V,\|\cdot\|)$ on a field F
  – is a pair $(V,\|\cdot\|)$ where V is a vector space and $\|\cdot\|$ a norm on V



- **Inner product space** $(V,\langle\cdot,\cdot\rangle)$ over the field F
  – is a vector space V over F together with an *inner product*:

$$\langle\cdot,\cdot\rangle:\ V\times V\to F\quad\text{such that:}$$
$$\langle x,y\rangle=\overline{\langle y,x\rangle}\qquad \forall x,y,z\in V\text{(conjugate symmetry)}$$
$$\begin{cases}\langle ax,y\rangle &= a\langle x,y\rangle\\ \langle x+y,z\rangle &= \langle x,z\rangle+\langle y,z\rangle\end{cases}\qquad \forall x,y,z\in V\text{(linearity in the first argument)}$$
$$\langle x,x\rangle>0,\quad x\in V\setminus\{\mathbf{0}\}\qquad \forall x,y,z\in V\text{(positive-definite)}$$

- **Complete metric space** (or Cauchy space)
  – every Cauchy sequence of points in the space converges to a point in the space

- **Banach space** $(V; \|\cdot\|)$ on a field F

  – is is a complete normed vector space

  $\rightarrow$ it has a metric that allows the computation of vector length and distance between vectors, and it is also complete in the sense that a Cauchy sequence of vectors always converges to a well defined limit that is within the space

Classical Banach spaces

	Dual space	Reflexive	weakly sequentially complete	Norm	Notes
$\mathbf{K}^n$	$\mathbf{K}^n$	Yes	Yes	$\|x\|_2 = \left(\sum_{i=1}^{n} \|x_i\|^2\right)^{1/2}$	Euclidean space
ℓ_p^n	ℓ_q^n	Yes	Yes	$\|x\|_p = \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}}$	
ℓ_∞^n	ℓ_1^n	Yes	Yes	$\|x\|_\infty = \max_{1\le i\le n} \|x_i\|$	
ℓ_p	ℓ_q	Yes	Yes	$\|x\|_p = \left(\sum_{i=1}^{\infty} \|x_i\|^p\right)^{\frac{1}{p}}$	
ℓ_1	ℓ_∞	No	Yes	$\|x\|_1 = \sum_{i=1}^{\infty} \|x_i\|$	
ℓ_∞	ba	No	No	$\|x\|_\infty = \sup_i \|x_i\|$	
$L^p(\mu)$	$L^q(\mu)$	Yes	Yes	$\|f\|_p = \left(\int \|f\|^p \, d\mu\right)^{\frac{1}{p}}$	
$L^1(\mu)$	$L^\infty(\mu)$	No	Yes	$\|f\|_1 = \int \|f\| \, d\mu$	The dual is $L^\infty(\mu)$ if μ is σ-finite.

- **Banach algebra** A over F

  – F is the real or complex numbers field (or a non-Archimedean complete normed field)

  – is an associative algebra and a Banach space ($\rightarrow$ multiplication is continuous)

  – is called *unital* if it has an identity element for the multiplication whose norm is 1

  – is called *commutative* if its multiplication is commutative.

  – any A can be embedded isometrically into a unital one A_e (A forms a closed ideal of A_e) [3]

- ***-ring** $(R; +, \cdot, ^*)$

  – is a ring with an operation $^* : R \rightarrow R$ that is an antiautomorphism and an involution, that is, it satisfies these properties:

$$\begin{aligned}
&(c+s)^* = r^* + s^* && \forall r, s \in R \\
&(r \cdot s)^* = s^* \cdot r^* && \forall r, s \in R \\
&1^* = 1 && \\
&(r^*)^* = r && \forall r \in R
\end{aligned}$$

  – also called *involutive* ring, involutory ring, or ring with involution

  – elements s.t. $x^* = x$ are called *self-adjoint*

  – examples: fields of complex or algebraic numbers with complex conjugation as involution

- ***-algebra** A over a ring R

  – consider commutative *-ring R

  – A is a *-ring that is an associative algebra over R, such that:

$$(r \cdot x)^* = r' \cdot x^* \qquad \forall r \in R, x \in A$$

  – R is often $\mathbb{C}$ (* is the complex conjugation) then * on A is conjugate-linear in R:

$$(\lambda x + \mu y)^* = \lambda' x^* + \mu' y^* \qquad \forall \lambda, \mu \in \mathbb{C} \;\; x, y \in A$$

[3] You can often assume a priori that A is unital: you develop the theory considering A_e and then you apply the outcome in A. However, this is not the case all the time. For example, one cannot define all the trigonometric functions in a Banach algebra without identity.

- ∘ *-homomorphism $f : A \to B$
  - algebra homomorphism compatible with the involutions of A and B, i.e., $f(a^*){=}f(a)^*\ \forall a \in A$
- **Banach *-algebra** A over field $\mathbb{C}$
  - is a Banach algebra over the $\mathbb{C}$ field with an involution operation ($^* : A \to A$) such that:

$$
\begin{aligned}
&(x+y)^* = x^* + y^* && \forall x, y \in A \\
&(\lambda x)^* = \bar{\lambda} x^* && \forall \lambda \in \mathbb{C}\ \ \forall x \in A \\
&(x \cdot y)^* = y^* \cdot x^* && \forall r, s \in A \\
&(x^*)^* = x && \forall r \in A
\end{aligned}
$$

  → is a *-algebra over field $\mathbb{C}$!
  - in most natural examples, the involution is isometric:

$$\|x^*\| = \|x\|$$

- Banach **B*-algebra** A on field $\mathbb{C}$
  - is a Banach *-algebra that satisfy the condition:

$$\|xx^*\| = \|x\|^2 \quad \forall x \in A \quad \text{(B*-condition)}$$

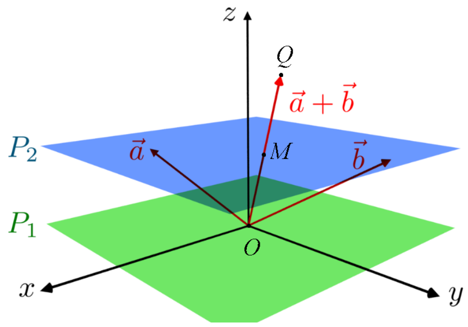
- Banach **C*-algebra** A over field $\mathbb{C}$
  - is a Banach *-algebra that satisfy the condition (adjoint property):

$$\|xx^*\| = \|x\|\,\|x^*\| \quad \forall x \in A \quad \text{(C*-condition)}$$

  → B*-algebra is also a C*-algebra and C*-condition implies the B*-condition:
  B*-algebra has been replaced by the term "C*-algebra"

Projective spaces:

- **Projective spaces**
  perspective: can be seen as a central projection of 3D space onto a plane
  pinhole camera model: pinhole = center of projection, image formed on the proj plane



  the center of projection is O
  the projection plane P_2 is a plane not passing through O (often chosen as the plane $z = 1$)
  consider a point Q which line OQ intersects P_2 in the point M
  the central projection (the "perspective") maps Q (in the "world") to $M \in P_2$ (the "image")

$$\exists M \Leftrightarrow Q \notin P_1$$

so we have two disjoint subsets:

(1) lines $\in P_1$ s.t. *one-to-one correspondence* with points $\in P_2$

(2) lines $\in P_1$ s.t. *no correspondence* with points $\in P_2$

so we can define:

– *projective points* of P_2 = *the lines* passing through O

– *projective line* = all projective points (which are lines) $\in$ a *plane* passing through O

the intersection of two planes passing through O = line passing through O

$\rightarrow$ intersection of two distinct projective lines = single projective point

(2) P_1 = projective line (*line at infinity* of P_2)

(1) *projective plane* is the disjoint union of P_2 and the (projective) line at infinity

- Projective space as an affine space (2^{nd} definition)

  – an affine space is like an Euclidean spaces but independent from the concepts of distance and measure of angles, keeping only the properties related to parallelism and ratio of lengths for parallel line segments.

  – a projective space can be seen as the associated vector space the preceding construction is generally done by starting from a vector space and is called projectivization.

- **Projective space** $P(V)$ over K (3^{rd} definition)

  – the set of equivalence classes of $V\backslash\{0\}$ under the equivalence relation $\sim$ that is:

$$v \sim w \quad \text{when } \exists \lambda \neq 0 \in K \text{ s.t.} \quad v = \lambda w$$

  – the equivalence classes for the relation $\sim$ are also called **(projective) rays**

Representation theory (RT)[4]:

I. RT studies algebraic struct $(A, \{\omega\})$ representing their **elements as lin. maps** L of vector spaces[5]

II. Then RT studies modules M over these $(A, \{\omega\})$

- A **representation** makes $(A, \{\omega\})$ more concrete by describing:

  – elements by matrices

  – algebraic operations as matrix addition and multiplication

- $(A, \{\omega\})$ can be groups, associative algebras and Lie algebras.

  The most prominent of these is the *representation theory of groups*:

  – elements of $(G, \star)$ are represented by invertible matrices s.t. $\star$ is a matrix multiplication

- The vector space V on which $(A, \{\omega\})$ is represented can be ∞-dimensional e.g. a Hilbert space so that , methods of analysis can be applied to the theory of groups

- RT is important in physics because, for example, describes how the symmetry group of a physical system affects the solutions of equations describing that system.

- RT led to numerous generalizations, as the *category theory* (CT): $(A, \{\omega\})$ as particular kinds of categories, and the representations as functors from the object category to the category of vector spaces.

[4]Theory of representations of algebraic structures by linear transformations and matrices (not to be confused with the presentation of a group).

[5] Linear map/transformation: a mapping $V \rightarrow W$ between two modules that preserves addition and scalar multiplication.

- Consider V a vector space over a field F. For $F = \mathbb{R}^n$ or $\mathbb{C}^n$ we can represent the elements with $n \times n$ matrices of real or complex numbers. This can be done with (1) groups, (2) associative algebras and (3) Lie algebras:

  (1) for the **general linear group** $GL_n(\mathbf{R})$ (= all invertible $n \times n$ matrices) that is closed under matrix multiplication, the elements are represented as invertible matrices $\rightarrow$ *RT of groups*

  (2) matrix addition + multiplication make $GL_n(\mathbf{R})$ an $\mathbf{R}$-algebra $\rightarrow$ *RT of associative algebras*

  (3) if in this $\mathbf{R}$-algebra we replace matrix multiplication MN by the matrix commutator $MN-NM$, it become a Lie algebra $\rightarrow$ *RT of Lie algebras*

  This can be generalized to any F and any V over F, with linear maps instead of matrices and composition instead matrix multiplication:

  (1) group $GL(V, F)$ of automorphisms of V

  (2) associative algebra $End_F(V)$ of all endomorphisms of V

  (3) corresponding Lie algebra $\mathfrak{gl}(V, F)$

- **Representation** Φ of G (1^{st} definition)[6]

  the representation of G (group, F-algebra or Lie algebra) G on V is a map

$$\Phi\colon G \times V \to V \quad (g, v) \mapsto \Phi(g, v)$$

  such that it sends g to this map, that has to be is linear over F $\forall g \in G$:

$$\Phi(g)\colon V \to V \qquad v \mapsto \Phi(g, v) := g \cdot v$$

  and such that, **for groups**:

$$\begin{cases} (1)\ \ e \cdot v = v & \forall v \in V, \quad e = \text{identity element of } G \\ (2)\ \ g_1 \cdot (g_2 \cdot v) = (g_1 g_2) \cdot v & \forall g_1, g_2 \in G, \quad g_1 g_2 = \text{product in } G \end{cases}$$

  For F-algebras, they don't always have an identity element, in which case equation (1) is ignored.

  Equation (2) is an abstract expression of the associativity of matrix multiplication. This doesn't hold for the matrix commutator and also there is no identity element for the commutator. Hence **for Lie algebras**:

$$(2')\ \ g_1 \cdot (g_2 \cdot v) - g_2 \cdot (g_1 \cdot v) = [g_1, g_2] \cdot v \quad \forall g_1, g_2 \in G, \quad \forall v \in V$$

  where $[g_1, g_2]$ is the Lie bracket, which generalizes the matrix commutator $MN - NM$

- **Representation** φ of G (2^{nd} definition)

  the map φ sends g in G to a linear map $\varphi(g) : V \to V$, which satisfies

$$\varphi(g_1 g_2) = \varphi(g_1) \circ \varphi(g_2) \quad \forall g_1, g_2 \in G$$

  From this point of view:

  - for a group, φ is a **group homomorphism** $\varphi : G \to GL(V, F)$
  - for a F-algebra, φ is an **algebra homomorphism** $\varphi : G \to End_F(V)$
  - for a Lie algebra, φ is a **Lie algebra homomorphism** $\varphi : G \to \mathfrak{gl}(V, F)$

[6] use the idea of *action*, generalizing how matrices act on column vectors during multiplication

4.2 Affine spaces

From a geometrical or a mechanical point of view, it is important to distinguish the affine Euclidian space E^3 from the underlying 3-dimensional linear space $\mathbf{R}^3$. An affine space is a linear space without a fixed origin. The elements of E^3 are points, while the elements of $\mathbf{R}^3$ are vectors. While E^3 is a bijective image of $\mathbf{R}^3$, it is not a vector space, and the elements of an affine space are not subject to the linear operations (vector addition and scalar multiplication).

An affine space is rigorously defined as follows, (Porteous 1981, Arnold 1979):

Definition. *Let X be a non-empty set and V a vector space. An* **affine structure** *on X with vector space V is a map*

$$c : X \times X \to V,\ c(x, v) = x - v,$$

which satisfies two axioms:

(i) *for all o in X the map* $c_o : X \to V,\ c_o(x) = x - o,$ *is bijective,*

(ii) *for all* x, y, o *in X,* $x - y = (x - o) - (y - o)$.

The set X, equipped with the affine structure is an **affine space**.

E^3 will denote a three-dimensional affine space over the linear space $\mathbf{R}^3$.

The inverse of the map c_o is denoted by $x = o + \mathbf{x}$, where $\mathbf{x} = x - o$. This defines the sum of a point, x, and a vector, $\mathbf{v}$. The result, $x + \mathbf{v}$, is the unique point y in E^3, such that $\mathbf{v} = y - x$.

By the use of reference frames (coordinate systems) the elements of both E^3 and $\mathbf{R}^3$ can be described by triples of real numbers. A coordinate system in E^3 is given by a point, o, in E^3 and a basis, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in $\mathbf{R}^3$. For a fixed choice of the reference frame, $o\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, every point, x, in E^3 is described by the coordinates of the vector $\mathbf{x} = x - o$ in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

E^3 is referred to as an affine *Euclidean space* when its vector space, $\mathbf{R}^3$, is a real orthogonal space with a positive-definite scalar product. The scalar product in $\mathbf{R}^3$ defines a *distance* function, d, on E^3.

Definition.

(1) *The* **distance** *between two points in* E^3 *is the function:*

$$d: E^3 \times E^3 \to \mathbf{R},\ d(x, y) = \sqrt{(x-y)\cdot(x-y)},$$

where "·" is the standard scalar product in $\mathbf{R}^3$.

(2) *The* **orientation** *function, o, is defined on frames* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *in* $\mathbf{R}^3$ *(or* E^3*), or, equivalently, on arrays of four points in* E^3, (o, x, y, z):

$$\sigma(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \begin{cases} 1 & \text{if } \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) > 0 \\ 0 & \text{if } \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) = 0 \\ -1 & \text{if } \det(\mathbf{x}, \mathbf{y}, \mathbf{z}) < 0 \end{cases}$$

The so-defined distance and orientation functions are, in general, dependant on the choice of a reference frame. This is so, since the scalar product and the determinant function are not invariant with respect to an arbitrary change of basis in $\mathbf{R}^3$. To ensure that both distance and orientation are invariant with respect to reference frame it is sufficient to restrict the choice of bases in $\mathbf{R}^3$ to only such triples $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ for which $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ and $\det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1$ (δ_{ij} is the Kronecker symbol). This assumption restricts the allowable coordinate systems in E^3. While the origin can be arbitrary, the coordinate vectors must satisfy the above conditions. Such reference frames are referred to as Cartesian.

4.3 The Euclidean group

It is assumed that a three-dimensional affine Euclidian space†, E^3, is given.

† All statements in this chapter can be made for a Euclidian space of arbitrary dimension. Of practical importance are mainly the cases $n = 2$ and $n = 3$. The theory for the plane ($n = 2$) can be derived from the spatial case ($n = 3$), since planar displacements are a subgroup of spatial displacements.

> ***Definition.*** *A* **displacement** *of E^3 is a transformation of E^3 (i.e., a map $g: E^3 \to E^3$), which preserves the distance and the orientation in E^3. The set of all displacements in three-dimensional space is denoted by $SE(3)$ and is known as the* **Euclidian group** *in three dimensions.*

It can be shown that a displacement is a bijective affine map and an automorphism (i.e., a homeomorphism of E^3 onto itself). The set $SE(3)$ is, indeed, a group. The group product of two displacements is defined as their composition as maps, $g_1 g_2(x) = g_1 \circ g_2(x) = g_1(g_2(x))$. The unit element of the group is the identity map on E^3, $e = id_{E^3}$. The inverse element of g is given by the inverse map, g^{-1}.

Example. A simple example of a displacement is the *translation* map, $g_t(x) = x + \mathbf{t}$, where $\mathbf{t}$ is a constant vector in $\mathbf{R}^3$. It maps a point x into the unique point y in E^3 such that $y - x = \mathbf{t}$. Since a translation is defined uniquely by a vector $\mathbf{t}$ the set of all translations, $\mathrm{Tr}(E^3)$, can be identified with $\mathbf{R}^3$ by means of the bijective map

$$\tau : \mathrm{Tr}(E^3) \to \mathbf{R}^3,\ \tau(g_t) = \mathbf{t}. \tag{2.1}$$

The addition of vectors in $\mathbf{R}^3$ turns the space of translations into an additive group. The unit element of this group is the translation by the zero vector, $\mathbf{o}$. In fact, τ is a group isomorphism between $\mathrm{Tr}(E^3)$ and $\mathbf{R}^3$. It can be seen that the group operation and the unit element in $\mathrm{Tr}(E^3)$ are identical with those in $SE(3)$, and therefore the group of translations is a sub-group of $SE(3)$.

Example. A displacement which maps at least one point, $o \in E^3$, into itself is called a *rotation*. The set of all rotations about an arbitrary fixed point o, $\mathrm{Rot}_o(E^3)$, is a subgroup of $SE(3)$. The group $\mathrm{Rot}_o(E^3)$ can be identified with $SO(3)$, the Special Orthogonal Group. $SO(3)$ consists of the so-called orthogonal linear maps in $\mathbf{R}^3$, i.e., the maps which preserve the scalar product and the determinant function. (When a Cartesian basis is fixed in $\mathbf{R}^3$, each element of $SO(3)$ is given by an orthogonal matrix with a positive determinant).

The isomorphism between $\mathrm{Rot}_o(E^3)$ and $SO(3)$ is given by the map

$$\rho: \mathrm{Rot}_o(E^3) \to SO(3),\ \rho(g) = G, \tag{2.2}$$

where $G : \mathbf{R}^3 \to \mathbf{R}^3$ is defined by

$$G\mathbf{x} = g(o + \mathbf{x}) - o. \tag{2.3}$$

The inverse map is $\rho^{-1}(F) = f$, where f is given by $f(x) = o + F(x - o)$.

To show that ρ is indeed an isomorphism, consider two rotations about o, f and g, and denote $fg = h$, $\rho(f) = F$, $\rho(g) = G$, $\rho(h) = H$. We need to show that $\rho(fg) = \rho(f)\rho(g)$, i.e., $H = FG$. From the definition of ρ (Equations 2.2 and 2.3), we have:

$$\begin{aligned} FG\mathbf{x} &= F(g(o + \mathbf{x}) - o) = f(o + (g(o + \mathbf{x}) - o)) - o = \\ &f(g(o + \mathbf{x})) - o = fg(o + \mathbf{x}) - o = h(o + \mathbf{x}) - o = H\mathbf{x}. \end{aligned} \tag{2.4}$$

The third equality in (2.4) follows from the definition of the operations "+" and "–" in the space E^3

Consider the set $\mathbf{R}^3 \times SO(3)$, which has as its elements the pairs of the type $(\mathbf{v}_f, F)$. We define a product operation by

$$(\mathbf{v}_f, F)(\mathbf{v}_g, G) = (F\mathbf{v}_g + \mathbf{v}_f, FG). \tag{2.5}$$

It can be shown that with this product $\mathbf{R}^3 \times SO(3)$ is a group with unit element $(\mathbf{o}, I)$, where I is the 3×3 unit linear map. This group will be denoted by $\mathbf{R}^3 \times_s SO(3)$. (Note: the symbol $G \times H$, where G and H are groups, denotes a group with a product operation different from the one in Equation (2.5), namely, $(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2)$.)

> ***Theorem.*** *(Arnold, 1980). The Euclidian group of all displacements in* E^3, $SE(3)$, *is isomorphic to the group* $\mathbf{R}^3 \times_s SO(3)$.

Proof. We will construct a map, $\phi : SE(3) \to \mathbf{R}^3 \times_s SO(3)$, and show that it is a group isomorphism.

First, for every displacement, f, and an arbitrary fixed point, o, we define, a translation, f_t, and a rotation, f_r, as:

$$f_t(x) = x + (f(o) - o), \qquad f_r(x) = f(x) + (o - f(o)). \tag{2.6}$$

The second equation in (2.6) defines a rotation, since $f_r(o) = o$.

We now define the map ϕ by:

$$\phi(f) = (\tau(f_t), \rho(f_r)). \tag{2.7}$$

The maps τ and ρ are the ones defined in Examples 2.1 and 2.2 (Equation (2.1) and Equations (2.2–3)), respectively. We denote

$$\mathbf{t}_f = \tau(f_t) = f(o) - o, \quad F = \rho(f_r). \tag{2.8}$$

We also consider the map $\psi : \mathbf{R}^3 \times SO(3) \to SE(3)$, defined as

$$\psi(\mathbf{t}_f, F) = \tau^{-1}(\mathbf{t}_f)\rho^{-1}(F). \tag{2.9}$$

The image of ψ is a displacement, f, which is a composition of the translation, $f_t = \tau^{-1}(\mathbf{t}_f)$, given by the vector $\mathbf{t}_f$, and the rotation, $f_r = \rho^{-1}(F)$, which corresponds to the orthogonal linear map F.

It can be checked that $\phi \circ \psi(\mathbf{t}_f, F) = (\mathbf{t}_f, F)$ and $\psi \circ \phi(f) = f$, i.e., $\psi = \phi^{-1}$. Thus, it is established that ϕ is bijective. This implies that every displacement, f, can be written as the product of a translation, f_t, and a rotation, f_r, which are obtained from f as shown in Equations (2.6).

It remains to show that the map ϕ preserves the group product, i.e., for any f, g in $SE(3)$, $\phi(fg) = \phi(f)\phi(g)$.

We denote $h = fg$ and $(\mathbf{t}_h, H) = \phi(fg)$. By the definition of the group product in $\mathbf{R}^3 \times_s SO(3)$ (Equation 2.5), we have: $\phi(f)\phi(g) = (\mathbf{t}_f, F)(\mathbf{t}_g, G) = (F\mathbf{t}_g + \mathbf{t}_f, FG)$. It must be proven that: (i) $\mathbf{t}_h = F\mathbf{t}_g + \mathbf{t}_f$, and (ii) $H = FG$:

(i) The definition of the translation vector, $\mathbf{t}_f$, and the orthogonal map of a displacement f (Equation 2.8) and the fact that $f = f_tf_r$ for every f, allow us to write the following sequence of equalities:

$$\begin{aligned}\mathbf{t}_h &= h(o) - o = fg(o) - o = fg_tg_r(o) - o = fg_t(o) - o = f_tf_rg_t(o) - o = (f_rg_t(o) + \mathbf{t}_f) - o = \\ &(f_rg_t(o) - o) + \mathbf{t}_f = ((o + F(g_t(o) - o)) - o) + \mathbf{t}_f = ((o + F\mathbf{t}_g) - o) + \mathbf{t}_f = F\mathbf{t}_g + \mathbf{t}_f.\end{aligned} \tag{2.10}$$

In (2.10), we also use the definitions of f_t and f_r in Equations (2.6).

(ii) Similarly to the proof of (i) above, we write:

$$\begin{aligned}H\mathbf{x} &= (h_r(o + \mathbf{x}) - o = (h(o + \mathbf{x}) + (-\mathbf{t}_h)) - o = (fg(o + \mathbf{x}) + (-\mathbf{t}_h)) - o = (f(g_tg_r(o + \mathbf{x})) + (-\mathbf{t}_h)) - \\ &(f(o + G\mathbf{x} + \mathbf{t}_g) - (-\mathbf{t}_h)) - o = ((f_r(o + G\mathbf{x} + \mathbf{t}_g) + \mathbf{t}_f) - (-\mathbf{t}_h)) - o = ((o + F(G\mathbf{x} + \mathbf{t}_g)) + (\mathbf{t}_f - \mathbf{t}_h)) - \\ &((o + FG\mathbf{x} + F\mathbf{t}_g) + (\mathbf{t}_f - F\mathbf{t}_g - \mathbf{t}_f)) - o = (o + FG\mathbf{x}) - o = FG\mathbf{x}.\end{aligned} \tag{2.11}$$

In the third-last equality in (2.11) we use the result of (i) (Eq. 2.10).

Thus, by proving (i) and (ii), it is established that ϕ is an isomorphism and the theorem is proven. □

4.4 Final remarks on $SE(3)$

⇨ Theorem *(Arnold, 1980)* → for a fixed choice of the origin, o, every displacement, g, of E^3 can be achieved in a unique way as a composition of a translation, g_t, and a rotation, g_r, $g = g_t g_r$. Moreover, every g is described by a pair $(\mathbf{t}_g, G)$, where G is an orthogonal linear map and $\mathbf{t}_g$ is a vector. The image of each point x with coordinates $\mathbf{x} = x - o$ is the point $g(x)$ with coordinates $G\mathbf{x} + \mathbf{t}$. Furthermore, it follows that each displacement is uniquely defined by a Cartesian coordinate system, $o_g\mathbf{e}_{g1}\mathbf{e}_{g2}\mathbf{e}_{g3}$, which is the image of the initial reference frame, $o\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The new origin, o_g, is the image of o under g, $o_g = g(o) = o + \mathbf{t}_g$, while the new coordinate vectors, $\mathbf{e}_{g1}$, $\mathbf{e}_{g2}$ and $\mathbf{e}_{g3}$, are the images of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ under the orthogonal map G, $\mathbf{e}_{gi} = G\mathbf{e}_i$.

⇨ The elements of $SE(3)$ are conveniently described by homogeneous 4×4 matrices of the type:

$$H_g = \begin{bmatrix} G & \mathbf{t}_g \\ 0 & 1 \end{bmatrix}.$$

The image of a point in E^3 under a displacement, g, with matrix H_g is obtained by pre-multiplying the column vector of the homogeneous coordinates of the point x, $(\mathbf{x}, 1)^T$, by the matrix: $H_g \begin{bmatrix} \mathbf{x} \\ 1 \end{bmatrix}$. The composition of two displacements is given by the matrix product.

⇨ Both the sets $\mathbf{R}^3$ and $SO(3)$ are at the same time three-dimensional smooth manifolds and groups under vector addition and matrix multiplication, respectively. The group $\mathbf{R}^3 \times_s SO(3)$ is, therefore, a smooth manifold of dimension six. The group operations in $SE(3)$ are smooth, since they are given by matrix multiplication of 4×4 matrices. Therefore, $SE(3)$ is a 6-dimensional Lie group which is a subgroup of $GL(4)$ (The group of nonsingular 4×4 matrices). $\mathrm{Tr}(E^3)$ and $\mathrm{Rot}_o(E^3)$ are 3-dimensional Lie subgroups of $SE(3)$.

4.5 A fast overview on Screw Theory (*article*)

Any rigid motion of 3d Euclidean space has a screw axis: a line mapped to itself (the fact that every element of the Euclidean group has at least one screw axis is called **Chasles' theorem**). We translate along this axis, and rotate about it.

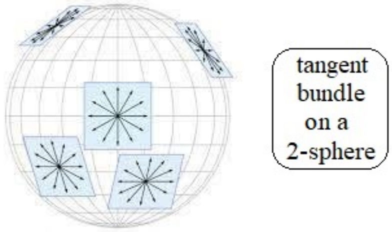
Screw theory is about the Euclidean group: the group of rigid motions of Euclidean space. A screw is an element of the Lie algebra of this group. It's a 6d vector built from a pair of 3d vectors: an infinitesimal translation and an infinitesimal rotation.

An object moving through space and rotating has a velocity and an angular velocity. These combine to form a screw. When you push on this object you exert a force and a torque on it. These also combine to form a screw. The screw combining velocity and angular velocity was called the twist, while the screw combining force and torque was called the wrench.

The Euclidean group is the semidirect product of the 3d rotation group $SO(3)$ and the translation group $\mathbb{R}^3$. Thus, we can write it as $SO(3) \ltimes \mathbb{R}^3$. It's isomorphic to $SO(3) \ltimes so(3)$, where we use the fact that any Lie group acts on its Lie algebra – which we can treat as a vector space, and thus an abelian Lie group. In fact any Lie group G acts on its Lie algebra $\mathfrak{g}$ and gives a Lie group $G \ltimes \mathfrak{g}$. This is isomorphic to the tangent bundle TG. So the tangent bundle of a Lie group is again a Lie group!

Putting all this together, the Euclidean group is isomorphic to the tangent bundle $TSO(3)$. A screw is an element of the Lie algebra of this! A less fancy way to say it: **screws live in $so(3) \ltimes \mathbb{R}^3$.**

The cross product of screws is the Lie bracket in $so(3) \ltimes \mathbb{R}^3$. The dot product is the obvious invariant inner product on this Lie algebra.



Screw theory was developed in the 1800s, and its terminology is charmingly mechanical. But in the 1800s, people preferred quaternions! So they had a different story.

We can think of $so(3)$ as the imaginary quaternions $ai + bj + ck$. Thus, we can think of $so(3) \ltimes \mathbb{R}^3$ as the imaginary quaternions tensored with the dual numbers $\mathbb{R}[\epsilon]/\langle\epsilon^2\rangle$. (Later, Grothenedieck and Lawvere thought of the dual numbers as the algebra of functions on an 'infinitesimal arrow'.)

Using these ideas, Clifford thought of screws as sitting inside the algebra of quaternions tensored with the algebra of dual numbers. He called this 8-dimensional algebra the dual quaternions (see Wikipedia, Dual quaternions). In other words, the dual quaternions are the algebra generated by i, j, k obeying the usual quaternion relations together with an element ϵ commuting with i, j, k and squaring to 0.

In the dual quaternions, the infinitesimal rotations are guys like $ai + bj + ck$, while the infinitesimal translations are guys like $\epsilon(ai + bj + ck)$. Together these form the 'screws'.

The screws are closed under commutators! They form the Lie algebra of the Euclidean group.

The dual quaternions can also be seen as the Clifford algebra of a real vector space with a quadratic form of signature $++0$, if we use the convention that the Clifford algebra on a vector space V with quadratic form Q is generated by $v \in V$ with relations $v_2 = -Q(v)$. The three generators are i, j and ϵk. But this is annoying asymmetrical! A better description is that the dual quaternions are the even part of the Clifford algebra of a vector space with quadratic form of signature $+++0$ (see Wikipedia, Clifford algebra: dual quaternions). I really like this, because the vector space with quadratic form of signature $+++0$ can be seen as the dual of Minkowski spacetime in the $c \to \infty$ limit.

Dual quaternions are still used in engineering, especially robotics. There will be even be a workshop about applications of dual quaternions to robotics at ICAR 2019, the International Conference on Advanced Robotics. Dan Piponi writes:

> They're also used in movie visual effects to simulate rigid-body dynamics. E.g. if you want to simulate a body thrown off a building without putting a stuntperson at risk.

I like the idea of the dual quaternions as an 'infinitesimal thickening' of the quaternions, and the Euclidean group $TSO(3)$ as an 'infinitesimal thickening' of the rotation group. Rogier Brussee writes:

> In algebraic geometry what you would do is consider the affine algebraic group $SO(V, g) \ltimes V$ which is defined by polynomial equations inside $End(V) \times V$ i.e. by an algebra $H = k[X_{ij}, T_i]/I$. The group structure is equivalent to a commutative but not cocommutative Hopf algebra structure $\Delta : A \to A \otimes A$ and $S : A \to A$.
>
> Let $\mathfrak{m}$ be the maximal ideal of the identity. Then we can consider the Hopf algebra $A/\mathfrak{m}^2$.
>
> This construction works for all algebraic groups, and gives an algebraic group $Spec(A/\mathfrak{m}2)$ with only one point and a non reduced structure (i.e. one that has nilpotent 'functions'). Apparently if V is 3-dimensional and we take the algebraic group $SO(V, g) \ltimes V$ the algebra of 'functions' splits over $k[\epsilon]/\langle\epsilon^2\rangle$!

– from The n-Category Café (https://golem.ph.utexas.edu/category/2019/10/screw_theory.html)

4.6 Rigid Body Motion

Summary from **Chapter 2** of *A Mathematical Introduction to Robotic Manipulation* by Richard M. Murray, Zexiang Li & S. Shankar Sastry

1. The *configuration* of a rigid body is represented as an element $g \in SE(3)$. An element $g \in SE(3)$ may also be viewed as a mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ which preserves distances and angles between points. In homogeneous coordinates, we write

$$g = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} R \in SO(3) \\ p \in \mathbb{R}^3. \end{matrix}$$

The same representation can also be used for a rigid body transformation between two coordinate frames.

2. *Rigid body transformations* can be represented as the exponentials of twists:

$$g = \exp(\widehat{\xi}\theta) \qquad \widehat{\xi} = \begin{bmatrix} \widehat{\omega} & v \\ 0 & 0 \end{bmatrix}, \qquad \begin{matrix} \widehat{\omega} \in so(3), \\ v \in \mathbb{R}^3, \theta \in \mathbb{R}. \end{matrix}$$

The twist coordinates of $\widehat{\xi}$ are $\xi = (v, \omega) \in \mathbb{R}^6$.

3. A twist $\xi = (v, \omega)$ is associated with a *screw* motion having attributes

$$\begin{aligned} &\text{pitch:} && h = \frac{\omega^T v}{\|\omega\|^2}; \\ &\text{axis:} && l = \begin{cases} \{\frac{\omega \times v}{\|\omega\|^2} + \lambda\omega : \lambda \in \mathbb{R}\}, & \text{if } \omega \neq 0 \\ \{0 + \lambda v : \lambda \in \mathbb{R}\}, & \text{if } \omega = 0; \end{cases} \\ &\text{magnitude:} && M = \begin{cases} \|\omega\|, & \text{if } \omega \neq 0 \\ \|v\|, & \text{if } \omega = 0. \end{cases} \end{aligned}$$

Conversely, given a screw we can write the associated twist. Two special cases are *pure rotation* about an axis $l = \{q + \lambda\omega\}$ by an amount θ and *pure translation* along an axis $l = \{0 + \lambda v\}$:

$$\xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} \theta \quad \text{(pure rotation)} \qquad \xi = \begin{bmatrix} v \\ 0 \end{bmatrix} \theta \quad \text{(pure translation).}$$

4. The *velocity* of a rigid motion $g(t) \in SE(3)$ can be specified in two ways. The *spatial velocity*,
$$\widehat{V}^s = \dot{g}g^{-1},$$
is a twist which gives the velocity of the rigid body as measured by an observer at the origin of the reference frame. The *body velocity*,
$$\widehat{V}^b = g^{-1}\dot{g},$$
is the velocity of the object in the instantaneous body frame. These velocities are related by the *adjoint transformation*
$$V^s = \mathrm{Ad}_g V^b \qquad \mathrm{Ad}_g = \begin{bmatrix} R & \widehat{p}R \\ 0 & R \end{bmatrix},$$
which maps $\mathbb{R}^6 \to \mathbb{R}^6$. To transform velocities between coordinate frames, we use the relations
$$\begin{aligned} V^s_{ac} &= V^s_{ab} + \mathrm{Ad}_{g_{ab}} V^s_{bc} \\ V^b_{ac} &= \mathrm{Ad}_{g^{-1}_{bc}} V^b_{ab} + V^b_{bc}, \end{aligned}$$
where V^s_{ab} is the spatial velocity of coordinate frame B relative to frame A and V^b_{ab} is the body velocity.

5. *Wrenches* are represented as a force, moment pair
$$F = (f, \tau) \in \mathbb{R}^6.$$
If B is a coordinate frame attached to a rigid body, then we write $F_b = (f_b, \tau_b)$ for a wrench applied at the origin of B, with f_b and τ_b specified with respect to the B coordinate frame. If C is a second coordinate frame, then we can write F_b as an *equivalent wrench* applied at C:
$$F_c = \mathrm{Ad}^T_{g_{bc}} F_b.$$
For a rigid body with configuration g_{ab}, $F^s := F_a$ is called the *spatial* wrench and $F^b := F_b$ is called the *body* wrench.

6. A wrench $F = (f, \tau)$ is associated with a screw having attributes
$$\text{pitch:} \qquad h = \frac{f^T\tau}{\|f\|^2};$$
$$\text{axis:} \qquad l = \begin{cases} \{\frac{f\times\tau}{\|f\|^2} + \lambda f : \lambda \in \mathbb{R}\}, & \text{if } f \neq 0 \\ \{0 + \lambda\tau : \lambda \in \mathbb{R}\}, & \text{if } f = 0; \end{cases}$$
$$\text{magnitude:} \qquad M = \begin{cases} \|f\|, & \text{if } f \neq 0 \\ \|\tau\|, & \text{if } f = 0. \end{cases}$$

7. A wrench F and a twist V are *reciprocal* if $F \cdot V = 0$. Two screws S_1 and S_2 are reciprocal if the twist V_1 about S_1 and the wrench F_2 along S_2 are reciprocal. The *reciprocal product* between two screws is given by
$$S_1 \odot S_2 = V_1 \cdot F_2 = V_1 \odot V_2 = v_1 \cdot \omega_2 + v_2^T\omega_1$$
where $V_i = (v_i, \omega_i)$ represents the twist associated with the screw S_i. Two screws are reciprocal if the reciprocal product between the screws is zero.

8. A *system of screws* $\{S_1, \ldots, S_k\}$ describes the vector space of all linear combinations of the screws $\{S_1, \ldots, S_k\}$. A *reciprocal screw system* is the set of all screws which are reciprocal to S_i. The dimensions of a screw system and its reciprocal system sum to 6 (in $SE(3)$).

All of the concepts presented in this chapter can also be applied to planar rigid body motions (see Exercises 10 and 11).

Related Bibliography The treatment of rigid motion described here, particularly the geometry of twists, was inspired by the work B. Paden (*Kinematics and Control Robot Manipulators.* PhD thesis, Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, 1986). The use of exponential coordinates for representing robotic motion was introduced by R. W. Brockett(“Robotic manipulators and the product of exponentials formula. In *P. A. Fuhrman, editor, Mathematical Theory of Networks and Systems*, pages 120?129. Springer-Verlag, 1984). Brockett?s derivation also forms the basis of the next chapter.

Related treatments can be found in the classical work by R. S. Ball (*A Treatise on the Theory of Screws.* Cambridge University Press, 1900) and the more recent texts by K. H. Hunt (*Kinematic Geometry of Mechanisms.* Oxford University Press, 1978), O. Bottema and B. Roth (*Theoretical Kinematics.* North-Holland, 1979), J. Duffy (*Analysis of Mechanisms and Robot Manipulators.* Edward Arnold Ltd., London, 1980), J. Angeles (*Rational Kinematics.* Springer-Verlag, 1988), and J. M. McCarthy (*An Introduction to Theoretical Kinematics.* MIT Press, 1990). A more abstract version of the developments of this chapter can be made in the framework of matrix Lie groups and is presented in Appendix A of this book (*A Mathematical Introduction to Robotic Manipulation* – Murray, Li, Sastry).

5 Kinetostatics of mechanisms

5.1 Planar twists and wrenches

5.2 Input/Output velocity equations

5.3 Statics & Principle of Virtual Work

5.4 Singularities

6 Assigments