ECN
–
S.I.P.R.O.
–
Summary Notes

Davide Lanza

2018-2019

Contents

Chapter 1

Signals

1.1 Continuous vs. Discrete

When "name"/"name" it refers continuous/discrete.

$x(t)$	Signal x	$(x[n])_{n\in\mathbb{Z}}$
Continuous-time signal is a function $x : \mathbb{R} \to \mathbb{C} \quad t \mapsto x(t)$	Definition	**Discrete-time signal** is a serie
$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	**Energy** of the signal x	$E_x = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$
$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \lvert x(t)\rvert^2 dt$	**Power** of the signal x	$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x(t)\rvert^2 dt$
$\overline{P_x} = \frac{1}{T} \int_{0}^{T} \lvert x(t)\rvert^2 dt$	**Mean power** of a periodic signal x (period $T \in \mathbb{R}$ or $N \in \mathbb{Z}$)	$\overline{P_x} = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x(t)\rvert^2 dt$
Energy signal $\Leftrightarrow E_x$ finite		**Power signal** $\Leftrightarrow P_x$ finite
$\underline{x}(t) = x(-t)$	**Time-reversed** signal	$\underline{x}[n] = x[-n]$

$$x^{(k)}(t) = \frac{dx}{dt}, \quad \boldsymbol{k < 0}$$

Derivative / **time-shift** of the signal x

$$x^{(k)}[n] = x[n+k], \quad k < 0$$

$$k < 0 \rightarrow (delay)$$

$$x^{(k)}(t) = \int_{-\infty}^{t} x^{(k)+1}(u)du, \quad \boldsymbol{k > 0}$$

Primitive / **time-shift** of the signal x

$$x^{(k)}[n] = x[n+k], \quad k > 0$$

$$k > 0 \rightarrow (pull\ ahead)$$

$$x_{a,t_o}(t) = \frac{1}{\sqrt{|a|}}\, x\Big(\frac{t-t_0}{a}\Big)$$

Transformation / **Interpolation** of the signal x

$$x_{N,n_0}[n] = \begin{cases} x\big[\frac{n-n_0}{N}\big] & if \;\; \frac{n-n_0}{N} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

$a > 1$ signal "stretched" on t axis. $1/\sqrt{|a|} \rightarrow$ **energy preserving** t_0 is a delay/pull-ahead.

is a **time-expanded**, add N 0-valued samples between each previous sample. It's **energy preserving**. n_0 is a delay/pull-ahead.

$$x_{a,t_o}(t) = \frac{1}{\sqrt{|a|}}\, x\Big(\frac{t-t_0}{a}\Big)$$

Transformation / **Decimation** of the signal x

$$x_{\frac{1}{N},n_0}[n] = [N(n-n_0)]$$

$a < 1$ signal "shrinked" on t axis.

is a **time-contracted**, remove all the samples between $\overline{n}$ and $\overline{n}+N$ samples. It's **<u>not</u> energy preserving**.

1.1.1 Examples

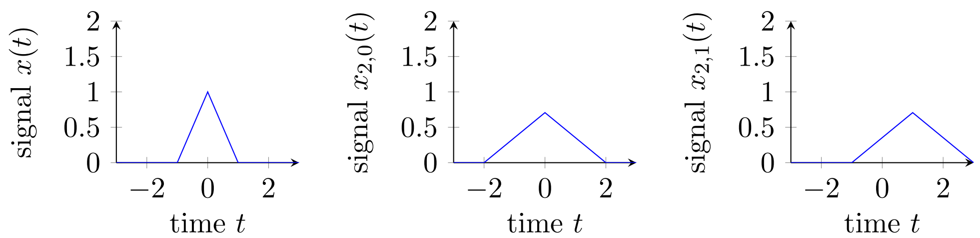


Figure 1.1: Transformation and time-shift (continuous-time signal)

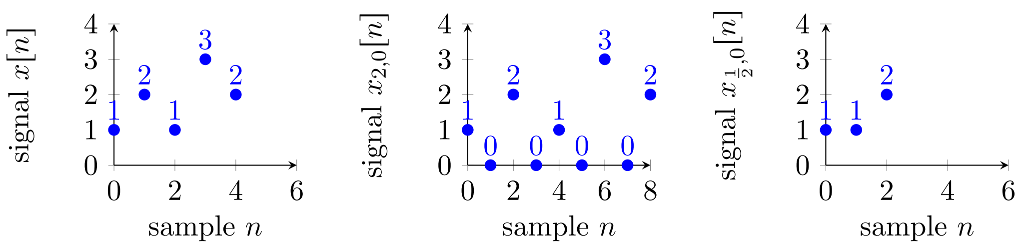


Figure 1.2: Interpolation and decimation (discrete-time signal)

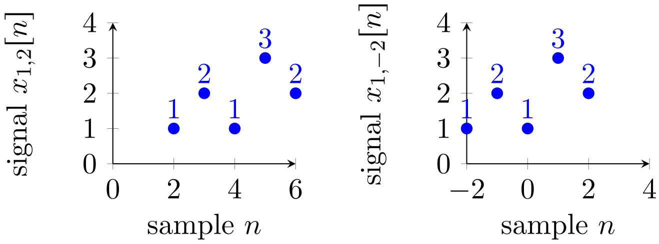
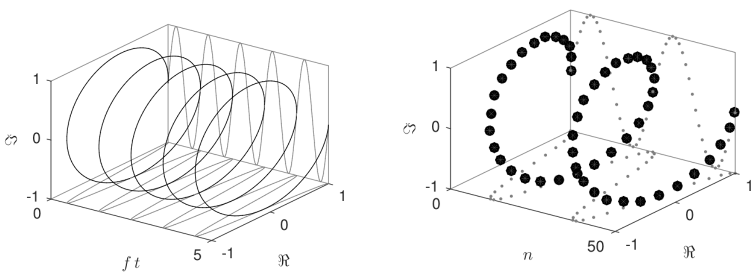


Figure 1.3: Delay and pull-ahead (discrete-time signal)

1.2 Standard Functions & Series

$\mathbf{1}(t) = 1$	**Unit constant**	$\mathbf{1}[n] = 1$
$t \in \mathbb{R} \mapsto a \cdot \exp\left[j(2\pi f t + \phi)\right]$	**Cisoid**	$\Big(a \cdot \exp\left[j(2\pi \lambda t + \phi)\right]\Big)_{n \in \mathbb{Z}}$
amplitude $a > 0$, frequency $f \in \mathbb{R}$, initial phase $\phi \in \mathbb{R}$		frequency $\lambda \in \mathbb{R}$ If $\exists k \in \mathbb{Z}$ $s.t$ $\lambda' = \lambda + k$ then: $a \exp\left[j(2\pi\lambda' t + \phi)\right] =$ $= a \exp\left[j(2\pi\lambda t + \phi)\right]$
$\text{step}(t) = \begin{cases} 0\,, & if\ \ t < 0 \\ 1\,, & if\ \ t \geqslant 0 \end{cases}$	**Step**	$\text{step}[n] = \begin{cases} 0\,, & if\ \ n < 0 \\ 1\,, & if\ \ n \geqslant 0 \end{cases}$
$[\delta] \underset{\boldsymbol{d/dt}}{\overset{\int_{-\infty}^{t}}{\rightleftarrows}} [\text{step}] \underset{\boldsymbol{d/dt}}{\overset{\int_{-\infty}^{t}}{\rightleftarrows}} [\text{ramp}]$		$\widetilde{\text{step}}[n] = \begin{cases} 0\,, & if\ \ n < 0 \\ \frac{1}{2}\,, & if\ \ n = 0 \\ 1\,, & if\ \ n > 0 \end{cases}$
$\text{ramp}(t) = \begin{cases} 0\,, & if\ \ t < 0 \\ t\,, & if\ \ t \geqslant 0 \end{cases}$	**Ramp**	$\text{ramp}[n] = \begin{cases} 0\,, & if\ \ n < 0 \\ t\,, & if\ \ n \geqslant 0 \end{cases}$

$\mathrm{rect}(t) = \begin{cases} 1\,, & if \quad -\frac{1}{2} < t < \frac{1}{2} \\ 0\,, & \text{otherwise} \end{cases}$ Integral and energy are 1.	**Rectangular window**	$\mathrm{rect}_N[n] = \begin{cases} 1\,, & if \;\; 0 \leqslant n \leqslant N-1 \\ 0\,, & \text{otherwise} \end{cases}$
$\delta(t) = \begin{cases} +\infty\,, & if \;\; t = 0 \\ 0\,, & \text{otherwise} \end{cases}$ More gen.: $t \mapsto \alpha\ \delta(t - t_0)$ α is the **mass** of the delta $\int_{-\infty}^{t} \alpha\ \delta(u)\ du = \alpha\ \mathrm{step}(t)$	**Dirac / Kroenecker delta** $\downarrow \downarrow \downarrow$	$\delta[n] = \begin{cases} 1\,, & if \;\; n = 0 \\ 0\,, & \text{otherwise} \end{cases}$
$\int_{-\infty}^{+\infty} x(t)\ \delta(t - t_0)\ dt = x(t_0)$	Sifting property	$\sum_{-\infty}^{\infty} x[n]\delta[n - n_0] = x[n_0]$
$\delta\left(\frac{t}{a}\right) = \lvert a \rvert\ \delta(t)$	Time scale for δ	
$\delta(t) = \int_{-\infty}^{+\infty} \exp[j\ 2\pi\ f\ t]\ df$	Integral formulation (Dirichlet?)	$\delta[n] = \int_{-1/2}^{+1/''} \exp[j\ 2\pi\ \lambda\ t]\ d\lambda$
$\mathbf{III}(t) = \sum_{-\infty}^{+\infty} \delta(t - k) =$ $= \sum_{-\infty}^{+\infty} \exp[j\ 2\pi\ f\ t]$	**Dirac / discrete-time comb**	$\mathbf{1}_{N,0}[n] = \sum_{-\infty}^{+\infty} \delta(n - kN) =$ $= \frac{1}{N} \sum_{0}^{N-1} \exp\left[j\ 2\pi\ \frac{k}{N}\ n\right]$ The comb is obtained interpolating the unit constant (that's why $\mathbf{1}_{N,0}[n]$).



(a) Cisoid (b) Discrete cisoid

Only continuous-time domain:

$$\operatorname{sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t} & (t \neq 0) \\ 1 & (t = 0) \end{cases} = \int_{-1/2}^{+1/2} \exp[j2\pi f t]\, df$$ **Cardinal sine** ×

Integral and energy are 1.
$\operatorname{sinc}(K) = 0,\ \forall K \in [\mathbb{Z} - \{0\}]$.
$\lim_{a \to 0} \frac{1}{|a|} \operatorname{sinc}\left(\frac{t}{a}\right) = \delta(t)$

↓ ↓ ↓

Time-contraction behavior

$$\operatorname{diric}_N(t) = \begin{cases} \frac{\sin(N\pi t)}{N \sin(\pi t)}, & if \ \ t \notin \mathbb{Z} \\ (-1)^{t(N-1)}, & if \ \ t \in \mathbb{Z} \end{cases}$$ **Dirichlet function** ×

Continuous function.
Periodic ($T = 1$) if N *odd*.
Periodic ($T = 2$) if N *even*,
but there is a symmetry w.r.t. $\left(\frac{1}{2}, 0\right)$.

↓ ↓ ↓

$$\operatorname{diric}_N\left(\frac{a}{N} \notin \mathbb{Z}\right) = 0,\ \forall a \in \mathbb{N}$$ Zeros

Main lobe around 0, the others are "side lobes".

$$D_N(t) = N \exp[-j\pi(N-1)t] \cdot \operatorname{diric}_N(t)$$ **Dirichlet kernel** ×

Periodic ($T = 1$) $\forall N$.

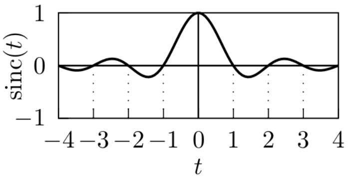
↓ ↓ ↓

$$D_N\left(\frac{a}{N} \notin \mathbb{Z}\right) = 0,\ \forall a \in \mathbb{N}$$ Zeros

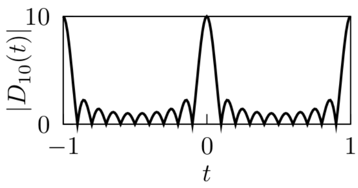
$$D_N(K) = N,\ \forall K \in \mathbb{Z}$$

$$\int_{-T/2}^{T/2} D_N(t)\, dt = 1 \ \ \Rightarrow \ \overline{P_x} = N$$ Mean power

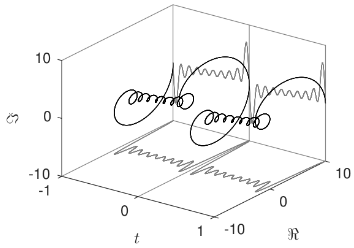
$$\lim_{N \to \infty} D_N(t) = \mathbf{III}(t)$$



(c) Cardinal sine



(d) Dirichlet function



(e) Dirichlet kernel

1.3 Convolution

$$(x * y)(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)\, d\tau \qquad \leftarrow \textbf{Definition} \rightarrow \qquad (x * y)[n] \sum_{k=-\infty}^{\infty} x[k])y[n-k]$$

– Properties:

Commutativity	Associativity	Identity element δ
$x * y = y * x$	$(x*y)*z = x*(y*x) = x*y*z$	$x * \delta = x$

Convolution with time-shifted pulse time-shifts
$x * \delta_{1,t_0} = x_{1,t_0}$

For periodic (T / N) signals:

$$(x \circledast y)(t) = \int_{0}^{T} x(\tau)y(t-\tau)\, d\tau \qquad \leftarrow \textbf{Definition} \rightarrow \qquad (x \circledast y)[n] \sum_{k=0}^{N-1} x[k])y[n-k]$$

– Properties:

Identity element (time-continuous)	Commutativity and Associativity	Identity element (time-discrete)
$\frac{1}{\sqrt{T}}\mathbf{III}_{T,0}$	(...)	$\mathbf{1}_{N,0}$

Convolution with time-shifted neutral element time-shifts

$\frac{1}{\sqrt{T}}x \circledast \mathbf{III}_{T,t_0} = x_{1,t_0}$ $\qquad$ $x \circledast \mathbf{1}_{N,n_0} = x_{1,n_0}$

1.4 Continuous → Discrete: Sampling

$$x_s[n] = x(t_n)_{n\in\mathbb{N}} \quad \left(\rightarrow\ t_n - t_{n-1} = T_s = f_s^{-1}\ \rightarrow\right) \quad x_s[n] = x(n\, T_s) = x(n/f_s)$$

If $x(t)$ has a discontinuity in $n_0\, T_s$, then $x[n_0] = x(n_0\, {T_s}^{+})$.

If $x(t)$ has a dirac pulse $\alpha\,\delta(t)$ in $n_0\,T_s$, then $x[n_0] = \frac{\alpha}{T_s}$.

So:

$$\begin{aligned}
\delta(t) &\overset{sampling}{\longrightarrow} \delta_s[n] = \tfrac{1}{T_s}\,\delta[n]\\
\mathrm{step}(t) &\overset{sampling}{\longrightarrow} \mathrm{step}_s[n] = \mathrm{step}[n]\\
\mathrm{ramp}(t) &\overset{sampling}{\longrightarrow} \mathrm{ramp}_s[n] = \tfrac{1}{T_s}\,\mathrm{ramp}[n]
\end{aligned}$$

1.5 Continuous ← Discrete: Holding

Impulse hold (**IH**)

$$x_{IH}(t) = T_s \sum_{n=-\infty}^{+\infty} x[n]\,\delta(t - n\,T_s)$$

Zero-order hold (**ZOH**)

$$\forall n \in \mathbb{Z},\ \forall t \in \big(nT_s, (n+1)nT_s\big),\ x_{ZOH}(t) = x[n]$$

→ → →

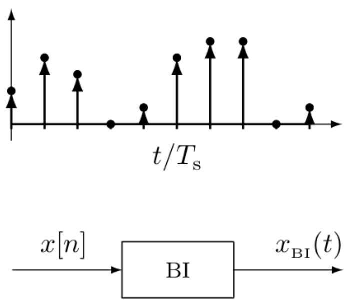
$$x_{ZOH}(t) = T_s \sum_{n=-\infty}^{+\infty} (x[n] - x[n-1])\ \mathrm{step}(t - n\,T_s)$$

First-order hold (**FOH**, linear interpolation)

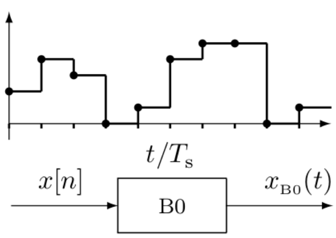
$$\forall n \in \mathbb{Z},\ \forall t \in \big(nT_s, (n+1)nT_s\big),$$
$$x_{FOH}(t) = x[n] + \Big((t - nT_s)\tfrac{x[n+1]-x[n]}{T_s}\Big)$$

→ → →

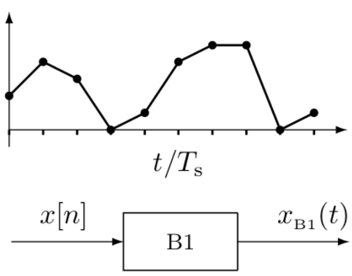
$$x_{FOH}(t) = T_s \sum_{n=-\infty}^{+\infty} (x[n{+}1]{-}2x[n]{+}x[n{-}1])\ \mathrm{ramp}(t{-}n\,T_s)$$



(f) Impulse Hold



(g) Zero-order Hold



(h) First-order Hold

1.6 Fourier Analysis

$$\left(\text{ Remind: } \quad \langle f|g\rangle = \int_{-\infty}^{\infty} f\cdot g^* \quad\right)$$

Fourier transform $\mathcal{F}_{cc} : x(t) \mapsto \mathcal{F}_{cc}x(f)$	**Fourier series** $\mathcal{F}_{cc}$	**Fourier transform** $\mathcal{F}_{dc} : x(t) \mapsto \mathcal{F}_{cc}x(\lambda)$	**DFT** $\mathcal{F}_{dd}$
$\underline{\boldsymbol{\mathcal{F}_{cc}}x(f) = \langle x(t) \vert e^{j\,2\pi\,f\,t}\rangle}$	$\underline{\boldsymbol{\mathcal{F}_{cd}}x[n] = \frac{1}{T}\langle x(t) \vert e^{j\,2\pi\,\frac{n}{T}\,t}\rangle}$	$\underline{\boldsymbol{\mathcal{F}_{dc}}x(\lambda) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\,2\pi\,\lambda\,n}}$	$\underline{\boldsymbol{\mathcal{F}_{dd}}x(k) = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j\,2\pi\,\frac{k}{N}\,n}}$
$x(t) = \langle \mathcal{F}_{cc}x(f) \vert e^{-j\,2\pi\,f\,t}\rangle$	$x(t) = \sum_{n=-\infty}^{\infty} \mathcal{F}_{cd}x[n]e^{j\,2\pi\,\frac{n}{T}\,t}$	$x[n] = \int_{-1/2}^{1/2} \mathcal{F}_{dc}x(\lambda)e^{j\,2\pi\,\lambda\,n}\,d\lambda$	$x[n] = \sum_{k=0}^{N-1} \mathcal{F}_{dd}x[k]e^{j\,2\pi\,\frac{k}{N}\,n}$
—	—	—	—
	$\mathcal{F}_{cc}x(f) = \sum_{n=-\infty}^{\infty} \mathcal{F}_{cd}x[n]\;\delta\!\left(f-\frac{n}{T}\right)$	$\mathcal{F}_{dc}x(\lambda) = \frac{1}{T_s}\,\mathcal{F}_{cc}x_{IH}\!\left(\frac{\lambda}{T_s}\right)$	$\mathcal{F}_{dc}x(\lambda) = \sum_{k=0}^{N-1} \mathcal{F}_{dd}x[k]\;\mathrm{III}\!\left(\lambda-\frac{k}{N}\right)$
Simmetry: $\mathcal{F}_{cc}x^*(f) = (\mathcal{F}_{cc}x(-f))^*$ $\mathcal{F}_{cc}\underline{x}(f) = \mathcal{F}_{cc}x(-f)$ $\mathcal{F}_{cc}\underline{x}^*(f) = (\mathcal{F}_{cc}x(f))^*$	(... More on symmetry after ...)	($\lambda = f/f_s$ is **dimensionless**!!!)	

1.6.1 Parseval theorem

Fourier transform preserves energy. If Ex finite:

$$(\boldsymbol{t}) \rightarrow \; E_x = \int_{-\infty}^{\infty} |x[n]|^2 = \int_{-\infty}^{\infty} |\mathcal{F}_{cc}x(f)|^2 \, df, \qquad [\boldsymbol{n}] \rightarrow \; E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-1/2}^{1/2} |\mathcal{F}_{dc}x(\lambda)|^2 \, d\lambda$$

$|\mathcal{F}_{cc}x(f)|^2 : f \mapsto |\mathcal{F}_{dc}x(f)|^2$ or $|\mathcal{F}_{dc}x(\lambda)|^2 : f \mapsto |\mathcal{F}_{dc}x(\lambda)|^2$ is the **energy spectrum** ($\rightarrow$ its integral over frequency gives the energy).

1.6.2 Symmetry properties

The following symmetry properties hold $\forall \mathcal{F}$:

Signal		Transform
Real	$x = x^*$	Real part is even, imaginary part is odd
Imaginary	$x = -x^*$	Real part is odd, imaginary part is even
Even	$x = \underline{x}$	Even

Signal		Transform
Odd	$x = -\underline{x}$	Odd
Even real part, odd imaginary part	$x = \underline{x}^*$	Real
Odd real part, even imaginary part	$x = -\underline{x}^*$	Imaginary

1.6.3 Fourier properties

	Fourier transform	**Fourier series**	**Fourier transform**	**DFT**
conv.		$\mathcal{F}_{cd}(x \circledast y) = \mathcal{F}_{cd}x \, \mathcal{F}_{cd}y$		$\mathcal{F}_{dd}(x \circledast y) = \mathcal{F}_{dd}x \, \mathcal{F}_{dd}y$
prod.	$\mathcal{F}_{cc}(x\,y) = \mathcal{F}_{cc}x * \mathcal{F}_{cc}y$	$\mathcal{F}_{cd}(x\,y) = \mathcal{F}_{cd}x * \mathcal{F}_{cd}y$	$\mathcal{F}_{dc}(x\,y) = \mathcal{F}_{dc}x \circledast \mathcal{F}_{dc}y$	$\mathcal{F}_{dd}(x\,y) = N\,\mathcal{F}_{dd}x \circledast \mathcal{F}_{dd}y$
sinus	if $x(t) = \mathrm{e}^{j\,2\pi\,f_0\,t}$ then $\mathcal{F}_{cc}x(f) = \delta(f - f_0)$	if $x(t) = \mathrm{e}^{j\,2\pi\,\frac{1}{T}\,t}$ then $\mathcal{F}_{cd}x[k] = \delta[k-1]$	if $x[n] = \mathrm{e}^{j\,2\pi\,\lambda_0\,n}$ then $\mathcal{F}_{dc}x(\lambda) = \mathrm{III}(\lambda - \lambda_0)$	if $x[n] = \mathrm{e}^{j\,2\pi\,\frac{1}{N}\,n}$ then $\mathcal{F}_{dd}x[k] = \frac{1}{N}\mathbf{1}_{N,0}[k-1]$
step	$\mathcal{F}_{cc}\,\mathrm{step}(f) = = \frac{1}{j\,2\pi\,f} + \frac{1}{2}\delta(f)$		$\mathcal{F}_{dc}\,\mathrm{step}(\lambda) = = \frac{1}{1-\mathrm{e}^{-j\,2\pi\,\lambda}} + \frac{1}{2}\mathrm{III}(\lambda)$	
comb	$\mathcal{F}_{cc}\,\mathrm{III} = \mathrm{III}$	$\mathcal{F}_{cd}\,\mathrm{III} = \mathbf{1}$	$\mathcal{F}_{dc}\,\mathbf{1}_{N,0} = \frac{1}{\sqrt{N}}\mathrm{III}_{1/N,0}$	$\mathcal{F}_{dd}\,\mathbf{1}_{N,0} = \frac{1}{N}\mathbf{1}$

For "sinus" $\rightarrow$ "modulation" and for time-scale see the Complex Transforms Properties (Section 1.7.2).

1.7 Complex analysis

1.7.1 Two-sided Laplace transform ($\nexists \mathcal{F}_{cc}$) vs. z-transform ($\nexists \mathcal{F}_{dc}$)

If $\mathcal{F}_{cc}x(f) = \int_{-\infty}^{\infty} x(t)e^{j\,2\pi\,f\,t}\,dt$ does not converge, $\nexists \mathcal{F}_{cc} \rightarrow$ we will use Laplace transform.

If $\mathcal{F}_{dc}x(\lambda) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\,2\pi\,\lambda\,n}$ does not converge, $\nexists \mathcal{F}_{dc} \rightarrow$ we will use z-transform.

Laplace transform
$\mathcal{L} : x(t) \mapsto \mathcal{L}x(s)$

z-transform
$\mathcal{Z} : x[n] \mapsto \mathcal{Z}x(z)$

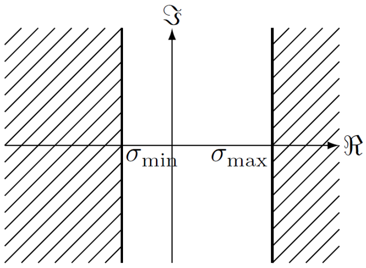
$$\mathcal{L}x(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

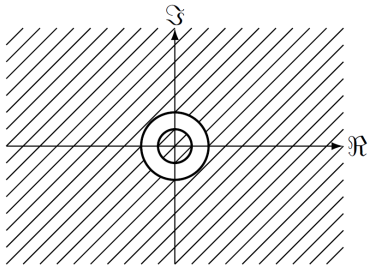
$$\mathcal{Z}x(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

$$\mathcal{L}x_{IH} = T_s\mathcal{Z}x(e^{-sT_s})$$

Convergence domain:
$\Sigma_x = \{s \mid \sigma_{min} < \Re\{s\} < \sigma_{max}\}$

Convergence domain:
$\Sigma_x = \{z \mid \rho_{min} < |z| < \rho_{max}\}$





(On boundaries ($s = \sigma$ or $|z| = \rho$) the transform can exist,
but we need theory of distributions to derive this transform properly)

For causal signals ($x(t) = 0\ \forall t < 0$))
$\rightarrow \Sigma_x \equiv$ right half-plane.
For anticausal signals ($x(t) = 0\ \forall t > 0$))
$\rightarrow \Sigma_x \equiv$ left half-plane.

For causal signals ($x[n] = 0\ \forall n < 0$))
$\rightarrow \Sigma_x \equiv$ outside a disk.
For anticausal signals ($x[n] = 0\ \forall n > 0$))
$\rightarrow \Sigma_x \equiv$ is a disk.

If $\Re = 0$ is included in Σ_x
$\rightarrow \mathcal{F}_{cc}x(f) = \mathcal{L}x(j\,2\pi\,f)$

If $|z| = 1$ is included in Σ_x
$\rightarrow \mathcal{F}_{dc}x(\lambda) = \mathcal{Z}x(e^{j\,2\pi\,\lambda})$

σ_{min} and σ_{max} are the real part
of poles of the Laplace transform.
There is **no pole** in Σ_x

ρ_{min} and ρ_{max} are the modulus
of poles of the z-transform.
There is **no pole** in Σ_x

Inverse Laplace transform	**Inverse z-transform**
For the inverse transform Σ_x is needed	For the inverse transform Σ_x is needed
Choose a σ s.t. $\{s \mid \Re\{s\} = \sigma\} \subset \Sigma_x$:	Choose a ρ s.t. $\{z \mid \lvert z\rvert = \rho\} \subset \Sigma_x$:
$x(t) = \int_{-\infty}^{\infty} \mathcal{L}x(\sigma + j2\pi f)\, \left[e^{(\sigma+j2\pi f)\, t}\right] df$	$x[n] = \int_{-1/2}^{1/2} \mathcal{Z}x\left(\rho e^{j\, 2\pi\, \lambda}\right) \left[\rho e^{j2\pi\lambda}\right]^n d\lambda$

The signal is decomposed as a sum of **damped cisoids**.

1.7.2 Complex transforms properties

Property:	**Laplace transform**:		**z-transform**:
– Linearity:	$\mathcal{L}(x+y) = \mathcal{L}x + \mathcal{L}y$	$[\Sigma_x \cap \Sigma_y] \subset \Sigma_{x+y}$	$\mathcal{Z}(x+y) = \mathcal{Z}x + \mathcal{Z}y$
	$\mathcal{L}(a\, x) = a\, \mathcal{L}x$	$\Sigma_{ax} \equiv \Sigma_x$	$\mathcal{Z}(a\, x) = a\, \mathcal{Z}x$
– Time-shift:	$\mathcal{L}x_{1,t_0}(s) = e^{-s\, t_0}\mathcal{L}x(s)$	$\Sigma_{x^{(k)}} \equiv \Sigma_x$	$\mathcal{Z}x^{(k)}(z) = z^k \mathcal{Z}x(z)$
– Modulation ($a \in \mathbb{C}$) :	$\downarrow\downarrow\ y(t) = x(t)e^{at}\ \downarrow\downarrow$		$\downarrow\downarrow\ y[n] = x[n]e^{an}\ \downarrow\downarrow$
	$\mathcal{L}y(s) = \mathcal{L}x(s-a)$	$\Sigma_y = \Sigma_x + a$	$\mathcal{Z}y(z) = \mathcal{Z}x(ze^{-a})$
→ *for Fourier*: (Modulation)	$\downarrow\downarrow\ a = j2\pi f_0\ \downarrow\downarrow$		$\downarrow\downarrow\ a = j2\pi\lambda_0\ \downarrow\downarrow$
	$\mathcal{F}_{cc}y(f) = \mathcal{F}_{cc}x(f - f_0)$		$\mathcal{F}_{dc}y(\lambda) = \mathcal{F}_{dc}x(\lambda - \lambda_0)$
– Derivative:	$\mathcal{L}\dot{x}(s) = s\, \mathcal{L}x(s)$ $\Sigma_x \subset \Sigma_{\dot{x}}$		
– Integral:	$\mathcal{L}x^{(-1)}(s) = \frac{1}{s}\, \mathcal{L}x(s)$ $[\Sigma_x \cap \{s \mid \Re(s) > 0\}] \subset \Sigma_{x^{(1)}}$		
– Time-scale:	$\mathcal{L}x_{a,0}(s) = \sqrt{\lvert a\rvert}\mathcal{L}x(as)$ $\Sigma_{x_{a,0}} \equiv \Sigma_x/a$		$\mathcal{Z}x_{N,0}(z) = \mathcal{Z}x(z^N)$ $\Sigma_{x_{N,0}} \equiv \Sigma_x^{1/N}$
→ *for Fourier*: (Time-scale)	$\mathcal{F}_{cc}x_{a,0}(f) = \left(\mathcal{F}_{cc}x(f)\right)_{\frac{1}{a},0}$		
– Convolution:	$\mathcal{L}(x * y) = \mathcal{L}x\mathcal{L}y$	$[\Sigma_x \cap \Sigma_y] \subset \Sigma_{x*y}$	$\mathcal{Z}(x*y) = \mathcal{Z}x\mathcal{Z}y$

1.8 Transforms summary

1.8.1 Continuous time

Laplace transforms ($\alpha \in \mathbb{C}$, $k \in \mathbb{N}^*$)

	$x(t) =$	$\mathcal{L}x(s) =$	$\Re(s) \in$
Dirac pulse	$\delta(t)$	1	$\mathbb{R}$
Step	$\mathrm{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$	$\frac{1}{s}$	$]0, +\infty[$
Ramp	$\mathrm{ramp}(t) = t\ \mathrm{step}(t)$	$\frac{1}{s^2}$	$]0, +\infty[$
Causal damped sinusoid ($\alpha \in \mathbb{R}$)	$\mathrm{e}^{-\alpha\, t}\ \cos(\omega\, t + \phi)\ \mathrm{step}(t)$	$\frac{(s+\alpha)\ \cos\phi - \omega\ \sin\phi}{(s+\alpha)^2 + \omega^2}$	$]-\alpha, +\infty[$
Causal damped cisoid	$\mathrm{e}^{-\alpha\, t}\ \mathrm{step}(t)$	$\frac{1}{s+\alpha}$	$]-\Re(\alpha), +\infty[$
Generalization	$\frac{1}{(k-1)!}\ t^{k-1}\ \mathrm{e}^{-\alpha\, t}\ \mathrm{step}(t)$	$\frac{1}{(s+\alpha)^k}$	$]-\Re(\alpha), +\infty[$
Anticausal damped cisoid	$-\mathrm{e}^{-\alpha\, t}\ \mathrm{step}(-t)$	$\frac{1}{s+\alpha}$	$]-\infty, -\Re(\alpha)[$
Generalization	$-\frac{1}{(k-1)!}\ t^{k-1}\ \mathrm{e}^{\alpha\, t}\ \mathrm{step}(-t)$	$\frac{1}{(s+\alpha)^k}$	$]-\infty, -\Re(\alpha)[$
Damped cisoid ($\Re(\alpha) > 0$)	$\mathrm{e}^{-\alpha\, \lvert t \rvert}$	$\frac{2\,\alpha}{\alpha^2 - s^2}$	$]-\Re(\alpha), \Re(\alpha)[$
Rectangular window	$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } \lvert t \rvert < \frac{1}{2} \\ \frac{1}{2} & \text{if } t = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} \frac{1}{s}\left(\mathrm{e}^{\frac{s}{2}} - \mathrm{e}^{-\frac{s}{2}}\right) & \text{if } s \neq 0 \\ 1 & \text{if } s = 0 \end{cases}$	$\mathbb{R}$

Fourier transforms (continous time) ($f_0 \in \mathbb{R}$)

	$x(t) =$	$\mathcal{F}_{cc}x(f) =$
Dirac pulse	$\delta(t)$	1
Step	$\mathrm{step}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 1 & \text{if } t > 0 \end{cases}$	$\frac{1}{j\,2\pi\,f} + \frac{1}{2}\ \delta(f)$
Rectangular window	$\mathrm{rect}(t) = \begin{cases} 1 & \text{if } \lvert t \rvert < \frac{1}{2} \\ \frac{1}{2} & \text{if } t = \pm\frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$	$\mathrm{sinc}(f)$
Cardinal sine	$\mathrm{sinc}(t) = \begin{cases} \frac{\sin(\pi\, t)}{\pi\, t} & \text{si } t \neq 0 \\ 1 & \text{si } t = 0 \end{cases}$	$\mathrm{rect}(f)$
Dirichlet function	$\mathrm{diric}_N(t) = \begin{cases} \frac{\sin(N\,\pi\, t)}{N\ \sin(\pi\, t)} & \text{if } t \notin \mathbb{Z} \\ (-1)^{t\,(N-1)} & \text{if } t \in \mathbb{Z} \end{cases}$	$\frac{1}{N} \sum_{k=0}^{N-1} \delta(f - \frac{N-1}{2} + k)$
Dirichlet kernel	$D_N(t) = N\,\mathrm{e}^{-j\,\pi\,(N-1)\,t}\ \mathrm{diric}_N(t) = \sum_{k=0}^{N-1} \mathrm{e}^{-j\,2\pi\,k\,t}$	$\sum_{k=0}^{N-1} \delta(f+k)$
Unit constant	1	$\delta(f)$
Cisoid	$\mathrm{e}^{j\,2\pi\,f_0\,t}$	$\delta(f - f_0)$
Comb	$Ш(t) = \sum_{k=-\infty}^{+\infty} \delta(t-k)$	$Ш(f)$

1.8.2 Discrete time

z-transforms ($\alpha \in \mathbb{C}$, $k \in \mathbb{N}^*$, $N \in \mathbb{N}^*$)

	$x[n] =$	$\mathcal{Z}x(z) =$	$\lvert z\rvert \in$
Pulse	$\delta[n] = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$	1	$\mathbb{R}$
Step	$\text{step}[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geqslant 0 \end{cases}$	$\dfrac{1}{1-z^{-1}}$	$]1, +\infty[$
Ramp	$\text{ramp}[n] = n\ \text{step}[n]$	$\dfrac{z^{-1}}{(1-z^{-1})^2}$	$]1, +\infty[$
Causal damped cisoid	$(-\alpha)^n\ \text{step}[n]$	$\dfrac{1}{1+\alpha\, z^{-1}}$	$]\lvert\alpha\rvert, +\infty[$
Generalization	$\binom{n+k-1}{k-1} (-\alpha)^n\ \text{step}[n]$	$\dfrac{1}{(1+\alpha\, z^{-1})^k}$	$]\lvert\alpha\rvert, +\infty[$
Anticausal damped cisoid	$-(-\alpha)^n\ \text{step}[-n-1]$	$\dfrac{1}{1+\alpha\, z^{-1}}$	$]-\infty, \lvert\alpha\rvert[$
Generalization	$(-1)^k \binom{-n-1}{k-1} (-\alpha)^{-n}\ \text{step}[-n-k]$	$\dfrac{1}{(1+\alpha\, z^{-1})^k}$	$]-\infty, \lvert\alpha\rvert[$
Damped cisoid ($\lvert\alpha\rvert < 1$)	$(-\alpha)^{\lvert n\rvert}$	$\dfrac{1}{1+\alpha\, z^{-1}} + \dfrac{1}{1+\alpha\, z} - 1$	$]\lvert\alpha\rvert, \dfrac{1}{\lvert\alpha\rvert}[$
Rectangular window	$\text{rect}_N[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } 0 \leqslant n \leqslant N-1 \\ 0 & \text{if } n > N-1 \end{cases}$	$\begin{cases} \frac{1-z^{-N}}{1-z^{-1}} & \text{if } z \neq 1 \\ N & \text{if } z = 1 \end{cases}$	$\mathbb{R}$

Fourier transforms (discrete time) ($\lambda_0 \in \mathbb{R}$, $N \in \mathbb{N}^*$)

	$x[n] =$	$\mathcal{F}_{\text{dc}}x(\lambda) =$
Pulse	$\delta[n] = \begin{cases} 0 & \text{if } n \neq 0 \\ 1 & \text{if } n = 0 \end{cases}$	1
Step	$\text{step}[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n \geqslant 0 \end{cases}$	$\dfrac{1}{1-\mathrm{e}^{-j\,2\pi\,\lambda}} + \dfrac{1}{2}\,Ш(\lambda)$
Rectangular window	$\text{rect}_N[n] = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } 0 \leqslant n \leqslant N-1 \\ 0 & \text{if } n > N-1 \end{cases}$	$D_N(\lambda)$
Unit constant	1	$Ш(\lambda)$
Cisoid	$\mathrm{e}^{j\,2\pi\,\lambda_0\,n}$	$Ш(\lambda - \lambda_0)$
Comb	$\mathbf{1}_{N,0}[n] = \begin{cases} 1 & \text{if } \frac{n}{N} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$	$\dfrac{1}{\sqrt{N}}\ Ш_{\frac{1}{N},0}(\lambda) = Ш(N\,\lambda)$

1.9 Sampling of a discrete-time signal. Shannon theorem

$x_s[n] = x(n\,T_s), \qquad \mathcal{F}_{cc}x(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt, \qquad \mathcal{F}_{dc}x_s(\lambda) = \sum_{n=-\infty}^{\infty} x_s[n]e^{-j2\pi\lambda t}$

Is it possible to recover the continuous time signal x from the sampled signal x_s?
In the spectral domain: is it possible to recover $\mathcal{F}_{cc}x$ from $\mathcal{F}_{dc}x$?

$$\mathcal{F}_{cc}x(f) \simeq \frac{1}{f_s}\mathcal{F}_{dc}x_s\left(\frac{f}{f_s}\right) \qquad \to \text{under what assumptions is not an approximation?}$$

We can write $\mathcal{F}_{dc}x_s$ with respect to $\mathcal{F}_{cc}x$: [1]

$$\mathcal{F}_{dc}x_s(\lambda) = \mathcal{F}_{dc}x_s\Big(\frac{f}{f_s}\Big) = f_s \sum_{n=-\infty}^{\infty} \mathcal{F}_{cc}x(f - n\ f_s)$$

→ so, $\mathcal{F}_{dc}x_s$ is the sum of $\mathcal{F}_{cc}x$ replicas shifted every f_s (is a **periodic** Fourier transform). So, if $f_{max} < \frac{f_s}{2}$ the spectrum is not distorted in the frequency band $[-\frac{f_s}{2}, \frac{f_s}{2}]$. But if $f_{max} > \frac{f_s}{2}$ the spectrum is distorted around half the sampling frequency. This is called **aliasing**.

Shannon theorem: Under the *Shannon condition*: $\mathcal{F}_{cc}x(f) = 0 \ \ \forall f \notin \Big[-\frac{f_s}{2}, \frac{f_s}{2}\Big]$ then:

$$\underline{\mathcal{F}_{cc}x(f) = \frac{1}{f_s}\mathcal{F}_{dc}x_s\Big(\frac{f}{f_s}\Big)} \ \ \forall f \in [-\frac{f_s}{2}, \frac{f_s}{2}]$$

In the time domain, this leads to the *Whittaker-Shannon interpolation formula*:

$$x(t) = \sum_{n\in\mathbb{N}} x_s[n]\ \mathrm{sinc}\Big(\frac{t - nT_s}{T_s}\Big)$$

⇒ $\frac{f_s}{2}$ is the Shannon/Nyquist frequency.

1.10 Decimation of a discrete-time signal

$x_{\frac{1}{N},0}[n] = x[N\ n]$

Is it possible to recover the signal x from the down-sampled signal $x_{\frac{1}{N},0}$?
In the spectral domain: is it possible to recover $\mathcal{F}_{dc}x$ from $\mathcal{F}_{dc}x_{\frac{1}{N},0}$?

We can write $\mathcal{F}_{dc}x_{\frac{1}{N},0}$ with respect to $\mathcal{F}_{dc}x$: [2]

$$\mathcal{F}_{dc}x_{\frac{1}{N},0}(N\ \lambda) = \frac{1}{N}\sum_{l=-\infty}^{\infty} \mathcal{F}_{dc}x\Big(\lambda - \frac{l}{N}\Big)$$

→ so, $\mathcal{F}_{dc}x_{\frac{1}{N},0}$ is the sum of $\mathcal{F}_{dc}x$ replicas shifted every $\frac{1}{N}$. This is periodic with period $\frac{1}{N}$.

"Decimation" theorem: Under the condition: $\mathcal{F}_{dc}x(\lambda) = 0 \ \ \forall\lambda \notin \Big[-\frac{1}{2N}, 1 - \frac{1}{2N}\Big]$ then:

$$\underline{\mathcal{F}_{dc}x(\lambda) = N\mathcal{F}_{dc}x_{\frac{1}{N},0}(N\lambda)} \ \ \forall\lambda \in [-\frac{1}{2N}, \frac{1}{2N}]$$

In the time domain → $x[n] = \sum_{k=-\infty}^{\infty} x_{\frac{1}{N},0}[k]\ \mathrm{sinc}\Big(\frac{n - kN}{N}\Big)$

⇒ $\frac{1}{2N}$ is the limit frequency.

[1] Proof on SIPRO Book.
[2] Proof on SIPRO Book.

Chapter 2

Systems

Chapter 2 - Systems

2.1 Definitions

- SYSTEM: $y = \mathcal{S}(u)$ $\left(\rightarrow \text{ IMPULSE/STEP RESPONSE: } \mathcal{S}(\delta) / \mathcal{S}(\text{step})\right)$
- SUPERPOSITION: $\mathcal{S}(u+v) = \mathcal{S}(u) + \mathcal{S}(v)$ } LINEARITY } LTI
- SCALING: $\mathcal{S}(au) = a\,\mathcal{S}(u)$
- t-INVARIANCE: if $\mathcal{S}(u) = y$, then $\mathcal{S}(u_{+,\tau}) = y_{+,\tau} = y(t-\tau)$
- CAUSALITY: $\mathcal{S}(u) = \mathcal{S}(u(\tau)), \{u(\tau) \text{ s.t. } \tau \leq t\}$ ($\geq$ → ANTICAUSAL)
- DIRECT FEEDTHROUGH (D.FTGH): $\mathcal{S}(u) = \mathcal{S}(u(t))$

2.2 LTI Systems

$\mathcal{S}(\delta) := h$; $\mathcal{S}(u) = h * u$; $\boxed{\text{CAUSALITY CONDITION}}$: $\begin{cases} \text{DISCRETE: } (\text{NO D.FTGH}) \wedge (h[\emptyset] = \emptyset) \\ \text{CONTINUOUS: } (h(t) \text{ CAUSAL}) \wedge (\nexists\, \delta^{(m)}(t) \text{ in } t = \emptyset) \end{cases}$

(NO D.FTGH) $\leftrightarrow$ $(\nexists\, \delta(t) \text{ in } t=\emptyset)$ ⇢ $\boxed{y(t) = \underbrace{\int_{-\infty}^{t_-} u(\tau)\,h(t-\tau)\,d\tau}_{(h*u \text{ till } t_-)} + \underbrace{a\,\delta(t)}_{(h*u \text{ for } t)}}$ (D.FTGH)

$\int_{-\infty}^{+\infty} |h(t)|\,dt < +\infty$ ⟵ $\boxed{\text{STABILITY CONDIT.}}$ ⟶ $\sum_m |h[m]| < +\infty$

$\left(\mathcal{L}y = \mathcal{L}h \cdot \mathcal{L}u \;;\; \mathcal{F}_{cc}\,y = \mathcal{F}_{cc}\,h \cdot \mathcal{F}_{cc}\,u\right)$ (THE MODULUS IS THE SYSTEM GAIN) $\left(\mathcal{Z}y = \mathcal{Z}h \cdot \mathcal{Z}u \;;\; \mathcal{F}_{dc}\,y = \mathcal{F}_{dc}\,h \cdot \mathcal{F}_{dc}\,u\right)$ (THE MODULUS...)

- FREQ. RESPONSE: (test signal:) $u(t) = a \cdot e^{j(2\pi f_0 t + \Phi)} \rightarrow y(t) = a \cdot |\mathcal{F}_{cc}\,h(f_0)| \cdot e^{j(2\pi f_0 t + \Phi + \arg(\mathcal{F}_{cc}\,h(f_0)))}$

(LTI with $h \in \mathbb{R} \Rightarrow |\mathcal{F}h|$ EVEN, $\arg(\mathcal{F}h)$ ODD)

↳ IN BODE: $G_{dB} = 20 \log_{10}(G/G_{ref})$

- DELAYS: if $\mathcal{F}_{cc}\,h = \underbrace{A(f)}_{\text{MODULUS}}\, e^{j\overbrace{\Phi(f)}^{\text{PHASE}}}$ then (PHASE DELAY) $\left[\tau_\Phi = -\frac{1}{2\pi}\frac{\Phi(f)}{f}\right]$ and (GROUP DELAY) $\left[\tau_g = -\frac{1}{2\pi}\frac{d\Phi}{df}(f)\right]$ {FUNCTIONS OF f: $\tau_\Phi(f)$, $\tau_g(f)$}

↳ IF $C_{is}(t) = e^{j2\pi f_0 t}$, $u(t) = a(t)\,C_{is}(t)$ ← (CISOID WITH A SLOWLY VARYING AMPLITUDE $a(t)$)

THEN $(h*u)(t) \approx A(f_0) \cdot a(t - \tau_g(f_0)) \cdot C_{is}(t - \tau_\Phi(f_0))$

16

2.3 Systems (LTI): Examples

- **Pure delay:**

$$\left[\begin{array}{lcl} y(t) = u(t-t_\emptyset) & ; & y[n] = u[n-n_\emptyset] \\ \downarrow & & \downarrow \\ \mathcal{L}\,h(s) = e^{-st_\emptyset} & ; & \mathcal{Z}\,h(z) = z^{-n_\emptyset} \end{array}\right]$$

$\}\ \mathcal{S}'(S)$

- **Pure gain:**

$$\left[\; y = G\,u \qquad \mathcal{L}\,h(s) = \mathcal{Z}(z) = G \;\right]$$

- **Differentiator:**

$$\left[\begin{array}{l} y = \dot{u} \to h(t) = \delta(t) \to \mathcal{L}\,h(s) = s \\ y = d^{(k)}u/dt^k \to h(t) = \delta^{(k)}(t) \to \mathcal{L}\,h(s) = s^k \end{array}\right]$$

- **Integrator:**

$$\left[\begin{array}{l} y(t) = \int_{-\infty}^{t} u(\tau)\,d\tau \to h(t) = \text{step}(t) \to \mathcal{L}\,h(s) = 1/s \\ \text{multiple integrator } (k) \to h(t) = \dfrac{t^k}{(k-1)!}\,\text{step}(t) \to \mathcal{L}\,h(s) = 1/s^k \end{array}\right]$$

- **F.O. sys.:**

$$\mathcal{L}\,h(s) = \frac{G}{s+\alpha} \quad (\alpha \in \mathbb{C}) \quad \text{2 sys}$$

$$\left[\begin{array}{l} h(t) = G\,e^{-\alpha t}\cdot\text{step}(t) \\ \text{RoC}: \{s \mid \text{Re}(s) > -\text{Re}(\alpha)\} \\ \text{STABLE IF } \text{Re}(-\alpha) < \emptyset \\ \text{CAUSAL SYSTEM} \end{array}\right] \quad \left[\begin{array}{l} h(t) = -G\,e^{\alpha t}\cdot\text{step}(-t) \\ \text{RoC}: \{s \mid \text{Re}(s) < -\text{Re}(\alpha)\} \\ \text{STABLE IF } \text{Re}(-\alpha) > \emptyset \\ \text{ANTICAUSAL SYSTEM} \end{array}\right]$$

$$\mathcal{Z}\,h(z) = \frac{G}{z+\alpha} = \frac{G z^{-1}}{1+\alpha z^{-1}} \quad \text{2 sys.}$$

$$\left[\begin{array}{l} h[n] = G(-\alpha)^{n-1}\,\text{step}[n-1] \\ \text{Roc}: \{z \mid |z| > |\alpha|\} \\ \text{STABLE IF } |-\alpha| < 1 \\ \text{CAUSAL SYSTEM} \end{array}\right] \quad \left[\begin{array}{l} h[n] = -G(-\alpha)^{n-1}\,\text{step}[-n] \\ \text{Roc}: \{z \mid |z| < |\alpha|\} \\ \text{STABLE IF } |-\alpha| > 1 \\ \text{ANTICAUSAL SYS} \end{array}\right]$$

- **Generalization:**

$$\mathcal{L}\,h(s) = \frac{1}{(s+\alpha)^k}$$

$$\left[h(t) = t^{k-1}/(k-1)!\; e^{-\alpha t}\,\text{step}(t)\right] \quad \left[h(t) = -t^{k-1}/(k-1)!\; e^{\alpha t}\,\text{step}(-t)\right]$$

$$\mathcal{Z}\,h(z) = \frac{1}{(z+\alpha)^k} = \frac{z^{-k}}{(1+\alpha z^{-1})^k}$$

$$\left[h[n] = \binom{n-1}{k-1}(-\alpha)^{n-k}\,\text{step}[n-k]\right] \quad \left[h[n] = (-1)^k\binom{-n+k-1}{k-1}(-\alpha)^{-n+k}\,\text{step}[-n]\right]$$

2.4 HOLDING : BLOCK DECOMPOSITION

HOLDING (SEE PG 7) SYS STRUCTURE:

- IN GENERAL: $x[m] \longrightarrow \boxed{\dfrac{1}{T_s\, \mathcal{Z}\phi_s(z)}} \to \dfrac{(x * \phi_s^{-1})[m]}{T_s} \to \boxed{\text{IMPULSE HOLD}} \to \dfrac{(x*\phi_s^{-1})_{IH}(t)}{T_s} \to \boxed{\mathcal{L}\phi(s)} \to x_H(t)$

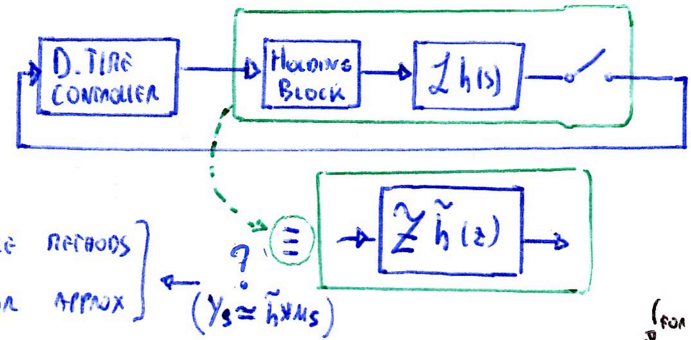
  $\left(\text{REMIND: } x_{IH}(t) = T_s \sum_m x[m]\,\delta(t - mT_s)\right)$ ‐ ‐ ‐ $\left[x_H(t) = \sum_m (x * \phi_s^{-1})[m]\,\phi(t - mT_s)\right]$

- $(\phi = \delta) \to$ IMPULSE HOLD ; $(\phi : \text{step}) \to$ ZOH ; $(\phi = \text{ramp}) \to$ FOH

  $\left(x_H * \phi^{-1} = \dfrac{1}{T_s}(x * \phi_s^{-1})_{IH}\right)$ $\left(\dfrac{\mathcal{L}x_H(s)}{\mathcal{L}\phi} = \dfrac{\mathcal{Z}x}{\mathcal{Z}\phi_s}(e^{sT_s})\right)$

2.5 System Sampling

[Control a continuous sys with a discrete controller]



[? → PERFECT EQUIVALENCE → INVARIANCE METHODS
? → APPROX. " " → INTEGRATOR APPROX]

- INVARIANCE METHODS:

  $\tilde{h} * \phi_s = (h * \phi)_s \quad \to \quad \mathcal{Z}\tilde{h}(z) = \dfrac{\mathcal{Z}(h*\phi)_s(z)}{\mathcal{Z}\phi_s(z)}$

  PERFECT EQUIVALENCE

  (FOR ϕ_s LOOK AT PG 7 !)

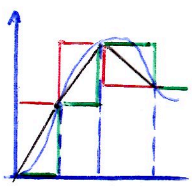
  $\begin{cases} \phi = \delta & (\text{IMPULSE INVARIANCE METHOD}) \\ \phi : \text{step} & (\text{STEP " " " "} \to \text{ZOH METHOD}) \\ \phi : \text{ramp} & (\text{RAMP " " " "} \to \text{FOH METHOD}) \end{cases}$

  For STEP INVARIANCE: $\left[\mathcal{Z}\tilde{h}(z) = \dfrac{\mathcal{Z}(h * \text{step})_s}{\mathcal{Z}\,\text{step}} = (1 - z^{-1})\,\mathcal{Z}(h * \text{step})_s(z)\right]$

  [PERFECT EQUIVALENCE FOR ZOH METHOD ⇕ INPUT IS CONSTANT BETWEEN THE TWO SAMPLING TIMES] [→ PG. 50 SUMMARY]

- INTEGRATOR APPROXIMATIONS:

  $$\mathcal{Z}\tilde{h}(z) \approx \frac{\mathcal{Z}y_s(s)}{\mathcal{Z}u_s(s)} = \begin{cases} T_s\, \mathcal{Z}\,\text{step}^{(-1)}(z) = T_s\; z^{-1}/1 - z^{-1} = 1/E_1(z) & (\text{left rectangle}) \\ T_s\, \mathcal{Z}\,\text{step}(z) = T_s\; 1/1 - z^{-1} = 1/E(z) & (\text{right rectangle}) \\ T_s\, \mathcal{Z}\,\widetilde{\text{step}}(z) = T_s/2\; 1 + z^{-1}/1 - z^{-1} = 1/B(z) & (\text{trapezoidal}) \end{cases}$$

  [THIS IS FOR $h(s) = 1/s$] → SO, IN GENERAL: $\mathcal{Z}\tilde{h}(z) \approx \begin{cases} \mathcal{L}h(E_1(z)) \\ \mathcal{L}h(E(z)) \\ \mathcal{L}h(B(z)) \end{cases}$ (WITH) $\begin{cases} E_1(z) = z - 1/T_s \\ E(z) = 1 - z^{-1}/T_s \\ B(z) = (2/T_s)\; 1 - z^{-1}/1 + z^{-1} \end{cases}$

2.6 LDEs

$$\sum_{i=0}^{M} b_{M-i}\, u^{(i)} = \sum_{j=0}^{N} a_{N-j}\, y^{(j)} \quad ; (b_M \neq 0 ;\ a_N \neq 0)$$

- CONTINUOUS: $y^{(k)}(t) = \frac{d^k y(t)}{dt^k}$
- DISCRETE: $y^{(k)}[n] = y[n+k]$

DERIVATIVE THEOREM / TIME-SHIFTING THEOREM

$$\mathcal{L}h(s) = \frac{\sum_{i=0}^{M} b_{M-i}\, s^i}{\sum_{j=0}^{N} a_{N-j}\, s^j} \qquad \mathcal{Z}h(z) = \frac{\sum_{i=0}^{M} b_{M-i}\, z^i}{\sum_{j=0}^{N} a_{N-j}\, z^j} \qquad (\forall s, z \in RoC)$$

For a CAUSAL SYSTEM:

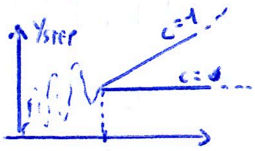
↳ $(N \geq M)$ ↳ (D.FTCH $\Leftrightarrow N = M$) ↳ (STABLE $\Leftrightarrow$ $Re(\text{poles}) < 0$ (continuous); $|\text{poles}| < 1$ (discrete))

$$\mathcal{L}h(s) = \frac{\sum_{i=\max(0,-c)}^{M-1}\left[b_{M-i}\, s^i\right] + b_0\, s^M}{\sum_{j=\max(0,c)}^{N-1}\left[a_{N-j}\, s^j\right] + s^N} = s^{-d}\, \frac{b_0 + \sum_{i=1}^{q}\left[b_i s^{-i}\right]}{1 + \sum_{j=1}^{p}\left[a_j s^{-j}\right]}$$

↳ c := SYSTEM CLASS ($c > 0 \rightarrow$ sys stable & causal) (# OF POLES IN $s=0$)

↳ STATIC GAIN := $K = \left[\lim_{s\to 0} s^c\, \mathcal{L}h(s)\right] \Rightarrow \left[\lim_{t\to+\infty} (h * step)^{(c)}(t) = K\right]$ STATIC VALUE OF c-th DERIVATIVE!

↳ IF $c = 0 \Rightarrow \left[\mathcal{L}h(0) = k\right]$



$$\mathcal{Z}h(z) = \frac{\sum_{i=\max(0,-c)}^{M-1}\left[b_{M-i}\, z^i\right] + b_0\, z^M}{\sum_{j=\max(0,c)}^{N-1}\left[a_{N-j}\, z^j\right] + z^N} = z^{-d}\, \frac{b_0 + \sum_{i=1}^{q}\left[b_i z^{-i}\right]}{1 + \sum_{j=1}^{p}\left[a_j z^{-j}\right]}$$

↳ d : PURE DELAY of sys ($u(n)$ impacts from $y(n+d)$)

↳ WE CAN RE-WRITE: $y[n] = -\sum_{j=1}^{p} a[j]\, y[n-k] + b_0\, u[n-d] + \sum_{i=1}^{q} b[i]\, u[n-d-i]$

2.7 System Response

If input & output values are not known before $t=\emptyset$ => NO SOLVE LDEs ↓ (DO THIS)

• Continuous:

(ONE SIDED LAPLACE) → $\mathcal{L}^+ y(s) = \int_{\emptyset^-}^{+\infty} y(t)\, e^{-st}\, dt$

CAUSAL SYS: $\mathcal{L}^+ \equiv \mathcal{L}$ & SAME PROPERTIES (→ PG 11) BUT ↓

- TIME SHIFT ONLY FOR FUTURE ($t_\emptyset > \emptyset$) OF CAUSAL SIGNAL (SEE PG 2) → TO FUTURE (DELAY) ↳ $[x_{1,t_0}(t) = x(t-t_0)]$ (PG 2)
- MODIFIED: DERIVATIVE THEOREM:

$$\mathcal{L}^+ \dot{y}(s) = s\,\mathcal{L}^+ y(s) + y(t=\emptyset^-)$$

$$\mathcal{L}^+ y^{(K)}(s) = s^K \mathcal{L}^+ y(s) - \left[\sum_{J=1}^{K} s^{K-J}\, y^{(J-1)}(s)\right]$$

- NEW: FINAL VALUE THEOREM:

$$\lim_{s\to\emptyset} s\,\mathcal{L}^+ y(s) = \lim_{t\to+\infty} y(t)$$

• Discrete:

(ONE SIDED Z-TRANSF.) → $\mathcal{Z}^+ y(z) = \sum_{n=\emptyset}^{+\infty} y[n]\, z^{-n}$

CAUSAL SYS: $\mathcal{Z}^+ \equiv \mathcal{Z}$ & SAME PROPERTIES (→ PG 11) BUT ↓

- TIME SHIFT NOT MODIFIED FOR $K<\emptyset$ (→ PG 2) → TO FUTURE (DELAY)
  (PG 2): $[x[n+K] = x_{1,-K}[n] = x^{(-K)}[n]]$
- MODIFIED: TIME SHIFT ($K>\emptyset$): → TO PAST (PULL AHEAD)

$$\mathcal{Z}^+ y^{(-1)}(z) = z^{-1}\,\mathcal{Z}^+ y(z) - y[-1]$$

$$\mathcal{Z}^+ y^{(-K)}(z) = z^{-K}\,\mathcal{Z}^+ y(z) - \left[\sum_{J=1}^{K} z^{-(K-J)}\, y[-J]\right]$$

↳ e.g.: ($K=3$) → $\left(z^{-3}\mathcal{Z} y(z) - z^{-2} y[-1] - z^{-1} y[-2] - y[-3]\right)$

(ONE SIDED $\mathcal{L}^+$ + MODIFIED DERIVATIVE → SOLVE LDE)

$$\mathcal{L}^+ y(s) = \underbrace{\mathcal{L} h(s)\, \mathcal{L}^+ u(s)}_{\text{RESPONSE OF THE CAUSAL PART OF THE INPUT}} + \frac{\sum_{K=\emptyset}^{N-1}\left(\sum_{\ell=K+1}^{N} a_{N-\ell}\, y^{(\ell-K+1)}(\emptyset^-)\right)s^K - \sum_{K=\emptyset}^{M-1}\left(\sum_{\ell=K+1}^{M} b_{M-\ell}\, u^{(\ell-K+1)}(\emptyset^-)\right)s^K}{\sum_{J=\emptyset}^{N-1}\left[a_{N-J}\, s^J\right] + s^N}$$

[SYS RESPONSE] [RESPONSE OF THE CAUSAL PART OF THE INPUT] [COMPLEMENTARY TERM WHICH DEPENDS ON $t=\emptyset^-$]

(ONE SIDED $\mathcal{Z}^+$ + MODIFIED SHIFTING ($K<\emptyset$) → SOLVE LDE) → [SAME CONCLUSION]

2.8 State Space Representation

$y = Cx + Du$; $\dot{x}(t) / x[n+1] = Ax + Bu$

(C: OUTPUT MATRIX) (D: DIRECT MATRIX) (A: STATE MATRIX / SYS. MATRIX) (B: INPUT MATRIX)

(NOT UNIQUE) → ($x_2 = Px$) → $\begin{cases} A_2 = PAP^{-1} \\ B_2 = PB \\ C_2 = CP^{-1} \\ D_2 = D \end{cases}$ (P SQUARE INVERTIBLE)

[A system with a SS. representation is LTI.]

↳ TRANSFER FUNCTION: $\left(\mathcal{L} h(s) = C(sI - A)^{-1}B + D \;;\; \mathcal{Z} h(z) = C(zI - A)^{-1}B + D\right)$

[CANONICAL FORMS] → (SEE PG. 18 COEFF.S)

↳

$$\underline{\underline{A}} = \begin{bmatrix} -a_1 (\dots) -a_{N-1} & -a_N \\ \underline{\underline{I}}_{N-1\times N-1} & \underline{\emptyset}_{N-1} \end{bmatrix} \quad \underline{B} = \begin{bmatrix} 1 \\ \underline{\emptyset}_{N-1} \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} \underline{\emptyset}_{N-M} & b_\emptyset (\dots) b_M \end{bmatrix} \quad D = \emptyset !$$

(DIRECT CANONICAL FORM)

[THESE FORMS ARE VALID IF ∄ D. TERM ! ↓ PG 57 SMD BOOK FOR THE COMPLETE ONE]

$$\underline{\underline{A}} = \begin{bmatrix} -a_1 & \\ (\vdots) & \underline{\underline{I}}_{N-1\times N-1} \\ -a_{N-1} & \\ -a_N & \underline{\emptyset}_{N-1} \end{bmatrix} \quad \underline{B} = \begin{bmatrix} \underline{\emptyset}_{N-M} \\ b_\emptyset \\ (\vdots) \\ b_M \end{bmatrix}$$

$$\underline{C} = \begin{bmatrix} 1 & \underline{\emptyset}_{N-1} \end{bmatrix} \quad D = \emptyset !$$

(INVERSE CANONICAL FORM)

CHARACTERISTIC POLINOMIAL: $y_A(s) = \det(sI - A) \to (\text{roots}) \to$ POLES $\leftarrow$ (eigenvalues of A) $\to \Delta = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_N \end{bmatrix}$

$y_A(s) = \sum_{i=0}^{N-1} [a_{N-i} s^i] + s^N \leftarrow$ (DENOMINATOR OF TRANS. FUNCTION)

SYSTEM RESPONSE: CONTINUOUS / DISCRETE

$\dot{x} = Ax + Bu$ SOLVE

$$\left[x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau\right]$$

(OMOGENEOUS SOLUTION) (VARIATION OF CONSTANTS)

$$e^{At} = \sum_{k=0}^{+\infty} \frac{1}{k!} (At)^k = e^{P \Delta P^{-1} t} = P\left[e^{\Delta t}\right] P^{-1}$$

$$\begin{bmatrix} e^{\lambda_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_N} \end{bmatrix}$$

OR

(FIND Δ BY $\det(sI - A)$)

OR

$\left(\mathcal{L}^{+\text{ inverse}}\left[(sI - A)^{-1}\right] = e^{At}\right)$

$x[n+1] = Ax[n] + Bu[n]$ SOLVE

$$\left[x[n] = A^{n-n_0} x[n_0] + \sum_{k=n_0}^{n-1} A^{n-k-1} B u[k]\right]$$

SYSTEM SAMPLING (SEE PG 18):

$$\begin{cases} y(t) = Cx(t) + Du(t) \\ \dot{x}(t) = Ax(t) + Bu(t) \end{cases} \xrightarrow{\text{SYSTEM SAMPLING}} \begin{cases} \tilde{y}[n] = \tilde{C}\tilde{x}[n] + \tilde{D} u_s[n] \\ \tilde{x}[n+1] = \tilde{A}\tilde{x}[n] + \tilde{B} u_s[n] \end{cases} \to [\text{STEP INVARIANCE}] \begin{cases} \tilde{x} = x_s \\ \tilde{A} = e^{AT_s} \\ \tilde{B} = \int_0^{T_s} e^{A\tau} B\, d\tau \\ \tilde{C} = C \\ \tilde{D} = D \end{cases}$$

↳ [RAMP INVARIANCE] → (PG 61 SIMMO BOOK) SAME FOR "TUSTIN TRANSFORM" (PG 62)

PG 63 → 67 : SIMPLE SYSTEM ANALYSIS ?